COMPLEX PROJECTIVE STRUCTURES WITH SCHOTTKY HOLONOMY

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Abstract. Let S be a closed orientable surface of genus at least two. Let Γ be a Schottky group whose rank is equal to the genus of S, and Ω be the domain of discontinuity of Γ. Pick an arbitrary epimorphism $\rho\colon \pi_1(S) \to \Gamma$. Then Ω/Γ is a surface homeomorphic to S carrying a (complex) projective structure with holonomy ρ. We show that every projective structure with holonomy ρ is obtained by (2π-)grafting Ω/Γ once along a multiloop on S.

1. Introduction

Let F be a connected orientable surface possibly with boundary. A **(complex) projective structure** is a $(\hat{\mathbb{C}}, \mathrm{PSL}(2,\mathbb{C}))$-structure, i.e. an atlas modeled on $\hat{\mathbb{C}}$, the Riemann sphere, with transition maps lying in $\mathrm{PSL}(2,\mathbb{C})$. It is well-known that a projective structure is equivalently defined as a pair (f, ρ) consisting of a topological immersion $f\colon \tilde{F} \to \hat{\mathbb{C}}$ (i.e. a locally injective continuous map), where $\tilde{F}$ is the universal cover of F, and $\rho\colon \pi_1(F) \to \mathrm{PSL}(2,\mathbb{C})$ is a homomorphism, such that f is ρ-equivariant, i.e. $f \circ \alpha = \rho(\alpha) \circ f$ for all $\alpha \in \pi_1(F)$ (see [19, §3.4]). The immersion f is called the **(maximal) developing map** and the homomorphism ρ is called the **holonomy (representation)** of the projective structure. A projective structure is defined up to an isotopy of F and an element of $\mathrm{PSL}(2,\mathbb{C})$, i.e. $(f,\rho) \sim (\gamma \circ f,\ \gamma \circ \rho \circ \gamma^{-1})$ for all $\gamma \in \mathrm{PSL}(2,\mathbb{C})$. If C is a projective structure on F, the pair (F, C) is called a **projective surface**. As usual, we will often conflate the projective structure C and the projective surface (F, C).

Throughout this paper, let S denote a closed orientable surface of genus at least two. The following theorem characterizes the holonomy representations of projective structures on S:

Theorem 1.1 (Gallo-Kapovich-Marden [3]). *A homomorphism $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ is a holonomy representation of some projective structure on S if and only if ρ satisfies: (i) the image of ρ is non-elementary and (ii) ρ lifts to a homomorphism from $\pi_1(S)$ to $\mathrm{SL}(2,\mathbb{C})$.*

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From the proof of this theorem, using (2π-)grafting operations, we can easily see that there are infinitely many projective structures with fixed holonomy ρ satisfying (i) and (ii) (see also [1]). Then it is a natural question to ask for a characterization of projective structures with the holonomy ρ. This question, at least, goes back to a paper by Hubbard ([6]; see also [3, §12]).

Basic examples of projective structures arise from Kleinian groups with nonempty domain of discontinuity. Let Γ be a (not necessarily classical) **Schottky group** of rank $g > 1$, which can be defined as a subgroup of $\mathrm{PSL}(2, \mathbb{C})$ isomorphic to a free group of rank g consisting only of loxodromic elements (for the details of Schottky groups, see [8, pp 75] [14]). Let $\Omega \subset \hat{\mathbb{C}}$ denote the domain of discontinuity of Γ. Let $\rho : \pi_1(S) \to \Gamma$ be an epimorphism (**Schottky holonomy (representation)**). Then Ω/Γ is a closed orientable surface S of genus g equipped with a canonical projective structure with holonomy ρ (with an appropriate marking on S given).

A grafting is an operation that transforms a projective structure into another projective structure without changing its base surface and holonomy representation (§3.5). It is a surgery operation that inserts a projective (actually affine) cylinder along an admissible loop, roughly speaking, a loop whose universal cover isomorphically embeds in $\hat{\mathbb{C}}$. If there is a multiloop consisting of disjoint admissible loops on a projective surface, a grafting operation can be done simultaneously along the multiloop.

A **Schottky structure** is a projective structure on S with Schottky holonomy. The goal of this paper is to prove the following theorem, which characterizes projective structures with fixed Schottky holonomy:

Theorem 8.1. *Every projective structure on S with Schottky holonomy ρ is obtained by grafting Ω/Γ once along a multiloop on S.*

Remark: Since ρ is quasiconformally conjugate to a representation from $\pi_1(S)$ onto a fuchsian Schottky group, the proof of Theorem 8.1 is reduced to the case that Γ is a fuchsian Schottky group, i.e. the limit set of Γ lies in the equator $\mathbb{R} \cup \{\infty\}$ of $\hat{\mathbb{C}}$ (c.f. [4]).

A projective structure is called **minimal** if it can *not* be otained by grafting another projective structure. Theorem 8.1 implies that Ω/Γ is the unique minimal structure among the projective structures on S with holonomy ρ, up to an element of a certain subgroup of the mapping class group of S (the orientation preserving part of this subgroup is $Stab_\rho$ defined below). There is an incorrect theorem in the literature

implying that there are many (essentially different) minimal structures with fixed Schottky holonomy (Theorem 3.7.3, Example 3.7.6 in [18]).

Theorem 8.1 is an analog to the case of a quasifuchsian holonomy: Let Γ' be a quasifuchsian group and let Ω^+, Ω^- be the connected components of the domain of discontinuity of Γ'. Then Ω^+/Γ' and Ω^-/Γ' are the projective surfaces on S, whose holonomy ρ' is an isomorphism from $\pi_1(S)$ onto Γ'.

Theorem 1.2 (Goldman [4])**.** *Every projective structure with quasifuchsian holonomy ρ' is obtained by grafting Ω^+/Γ' or Ω^-/Γ' along a multiloop.*

In Theorem 1.2, for a given projective structure, the choice of the multiloop and the basic structure, Ω^+/Γ' or Ω^-/Γ', is unique (up to the isotopy of the multiloop on S). On the other hand, in Theorem 8.1, there are infinitely many choices of the multiloop, which induce different markings on Ω/Γ.

Below, we shall discuss an approach to formulate a uniqueness theorem, generalizing Theorem 8.1. Fix an appropriate marking (and, therefore, an orientation) on Ω/Γ so that, with this marking, Ω/Γ is a projective structure with the holonomy ρ. Let $\mathcal{P}_\rho$ denote the collection of all projective structures on S with the Schottky holonomy ρ and the same orientation as that of Ω/Γ. Let $\phi\colon S \to S$ be a mapping class. Then the **support** of ϕ is the minimal subsurface R of S such that the restriction of ϕ to $S \setminus R$ is the identity map. The mapping class ϕ induces an automorphism $\phi^*\colon \pi_1(S) \to \pi_1(S)$. Let $Stab_\rho$ denote the subgroup of the mapping class group of S consisting of orientation-preserving mapping classes $\phi : S \to S$ such that $\rho \circ \phi^* = \rho$. It is known that $Stab_\rho$ is generated by Dehn twists along the loops on S that belong to $\ker(\rho)$ (see [12]). Let $\mathcal{AML}_\rho(S)$ denote the set of isotopy classes of multiloops on S consisting of disjoint admissible loops on Ω/Γ.

Conjecture 1.3. *Every $C \in \mathcal{P}_\rho$ can be obtained by changing the marking of Ω/Γ by a unique $\phi \in Stab_\rho$ and grafting Ω/Γ along a unique $L \in \mathcal{AML}_\rho(S)$ such that L and the support of ϕ are disjoint, where the support of ϕ is the minimal subsurface of S on which ϕ acts non-trivially:*

$$\mathcal{P}_\rho(S) \cong \{(\phi, L) \in Stab_\rho \times \mathcal{AML}_\rho(S) \mid Supp(\phi) \cap L = \emptyset\}.$$

Theorem 8.1 ensures that, for every $C \in \mathcal{P}_\rho$, there is a corresponding pair $(\phi, L) \in Stab_\rho \times \mathcal{AML}_\rho(S)$, but $Supp(\phi) \cap L$ might be nonempty. The conjecture above claims that this intersection can be uniquely "resolved".

Outline of the proof of Theorem 8.1. Fix a projective structure C on S with Schottky holonomy ρ. First we decompose (S, C) into certain very simple projective structures, called *good holed spheres* (§3.4), by cutting S along a multiloop M (Proposition 6.1). Note that an arbitrary region in $\hat{\mathbb{C}}$ is equipped with a canonical projective structure. Then each good holed sphere is obtained by grafting a holed-sphere F isomorphically embedded in $\hat{\mathbb{C}}$ along a multiarc properly embedded in F (Proposition 7.5). The multiloop on S in Theorem 8.1 is realized as the union of such multiarcs on the components of $S \setminus M$.

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2. Conventions and terminology

We follow the following conventions and terminology, unless otherwise stated:

- A surface is connected and orientable.
- A component is connected.
- A loop and arc are simple.
- A 1-dimensional manifold properly embedded in a surface is called a **multiloop** if it is a disjoint union of loops and a **multiarc** if a disjoint union of arcs.
- A loop on a surface F may represent an element of $\pi_1(F)$.
- Let F be a surface and let F_1, F_2 be subsurfaces of F. Then F_1 and F_2 are called **adjacent** if F_1 and F_2 share exactly one boundary component and have disjoint interiors.

3. Preliminaries

3.1. **Minimal developing maps.**

Let F be a surface possibly with boundary and let $\bar{F}$ be the universal cover of F. Let $C = (\bar{f}, \rho)$ be a projective structure on F. Then the short exact sequence

$$1 \to \ker(\rho) \to \pi_1(F) \xrightarrow{\rho} Im(\rho) \to 1$$

induces an isomorphism $\tilde{\rho}\colon \pi_1(F)/\ker(\rho) \to Im(\rho)$. Let $\tilde{F} = \bar{F}/\ker(\rho)$, which we call the **minimal cover** of F associated with ρ. Then, via $\tilde{\rho}$, $Im(\rho)$ acts on $\tilde{F}$ freely and properly discontinuously, and we have $\tilde{F}/Im(\rho) = F$. Define the **(minimal) developing map** $f\colon \tilde{F} \to \hat{\mathbb{C}}$ of

C to be the locally injective map satisfying $\bar{f} = \phi \circ f$, where $\phi : \bar{F} \to \tilde{F}$ is the canonical covering map.

by $f(x) = \bar{f}(\bar{x})$ for $x \in \tilde{F}$, where $\bar{x}$ is a lift of x to $\bar{F}$. It is easy to see that $f(x)$ does *not* depend on the choice of $\bar{x}$. Conversely, a $\tilde{\rho}$-equivariant immersion $f\colon \tilde{F} \to \hat{\mathbb{C}}$ lifts to a ρ-equivariant immersion $\bar{f}\colon \bar{F} \to \hat{\mathbb{C}}$. Therefore the projective structure C can be defined as the pair (f, ρ) consisting of the minimal developing map and the holonomy representation. For the remainder of the paper, we use this new pair (f, ρ) to represent a projective structure, and a developing map is always a minimal developing map, unless otherwise stated. For a projective structure C, we let $dev(C)$ denote its minimal developing map.

3.2. **Restriction of projective structures to subsurfaces.** Let $C = (f, \rho)$ be a projective structure on a surface F. Let E be a subsurface of F. The **restriction** of C to E is the projective structure on E given by restricting the atlas of C on F to E, and we denote the restriction by $C|E$. We can equivalently define $C|E$ as a pair of the (minimal) developing map and a holonomy representation as follows: The inclusion $E \subset F$ induces a homomorphism $i^*\colon \pi_1(E) \to \pi_1(F)$. Let $\tilde{E}$ be a lift of E to $\tilde{F}$ invariant under $\pi_1(E)$. Then $C|E$ is the projective structure on E given by $(f|_{\tilde{E}},\ \rho \circ i^*)$.

3.3. **Basic projective structures.** Let $C = (f, \rho)$ be a projective structure (on a surface). Then C is called **basic** (also called uniformizable) if the minimal developing map f is an homeomorphism onto a subset of $\hat{\mathbb{C}}$. This definition is equivalent to saying that the maximal developing map of C is a covering map onto a region in $\hat{\mathbb{C}}$.

The following is immediate:

Lemma 3.1. *Let $\rho\colon \pi_1(S) \to \Gamma \subset \mathrm{PSL}(2, \mathbb{C})$ be Schottky holonomy and C be a projective structure on S with holonomy ρ. Then C is a basic projective structure if and only if $dev(C)$ is a homeomorphism onto the domain of discontinuity of Γ.*

3.4. **Good and almost good projective structures.** Let F be $\mathbb{S}^2$ with finitely many disjoint points and disks removed, i.e. a genus-zero surface of finite type. Then ∂F is the union of the boundary components of the removed disks. Let P_F denote the points removed, i.e, the punctures of F.

A projective structure $C = (f, \rho)$ on F is **almost good** if it satisfies the following conditions:
(i) $\rho\colon \pi_1(F) \to \mathrm{PSL}(2, \mathbb{C})$ is the trivial representation ρ_{id} (and therefore

the domain of f is F);
(ii) f continuously extends to P_F, so that P_F is the set of ramification points of the extension;
(iii) there exists a surface R of finite type embedded in $\hat{\mathbb{C}}$ such that, via (the extension of) f, each point of P_F maps to a puncture of R and each component of ∂F covers a component of ∂R, and
(iv) $f(P_F) \cup f(\partial F)$ is *not* connected.

Note that (iv) implies that the Euler characteristic of F is non-positive. The surface $R \subset \hat{\mathbb{C}}$ is called a **support** of the almost good projective structure C. The support R can be chosen uniquely so that $P_R = f(P_F)$ and $\partial R = f(\partial F)$, where P_R is the set of the punctures of R (for a general support, we only have $P_R \supset f(P_F)$ and $\partial R \supset f(\partial F)$). This unique support is called the **full support** of the almost good structure $C = (f, \rho_{id})$ and denoted by $Supp(C)$ or, alternatively, $Supp_f(F)$. Note that Condition (iii) implies that f has the lifting property along every path p on $\hat{\mathbb{C}}$ such that p is disjoint from the punctures of R and p does *not* cross the boundary components of R.

A projective structure $C = (f, \rho)$ on F is **good** if it satisfies Conditions (i), (ii), (iii), (vi) and, in addition,
(v) there is a bijective correspondence, via f, between the punctures and boundary components of F and those of R.

Assume that C is a good structure on F. Then, by (v), R is the full support of C. Thus there is a basic projective structure $C_0 = (f_0, \rho_{id})$ such that f_0 is a homeomorphis from F to R and $f_0(\ell) = f(\ell)$ for every puncture and boundary component ℓ of F. We call C_0 a **basic structure associated with** C. Note that C_0 is unique up to the marking on F, in other words, an element of the pure mapping class group of F.

Now let us return to the case that $C = (f, \rho_{id})$ is an almost good projective structure on F supported on $R \subset \hat{\mathbb{C}}$. Let ℓ' be a boundary component of R and let ℓ_i ($i = 1, 2, \ldots, n$) be the boundary components of F that cover ℓ' via f. Let $\hat{F}$ be the surface obtained from F by attaching a once-punctured disk along each ℓ_i (topologically, we pinch $\ell_1, \ell_2, \ldots, \ell_n$ into punctures). In the following, we extend the almost good structure C on F to an almost good structure on $\hat{F}$. Let D' be the component of $\hat{\mathbb{C}} \setminus R$ bounded by ℓ'. Then topologically D' is a disk. Up to an element of $\mathrm{PSL}(2, \mathbb{C})$, we can assume that D' is a bounded region in $\mathbb{C}$. For each $i \in \{1, 2, \ldots, n\}$, let d_i denote the degree of the covering map $f|_{\ell_i} \colon \ell_i \to \ell'$. Then pick a point p'_i in $int(D')$ and define $\phi_i \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ by $\phi_i(z) = (z - p'_i)^{d_i} + p'_i$. Let D_i denote $\phi_i^{-1}(D') \setminus \phi_i^{-1}(p'_i) = \phi_i^{-1}(D') \setminus p'_i$, which is a once-punctured disk. Then $(\phi_i|_{D_i}, \rho_{id})$ is a good

projective structure on D_i, where $\rho_{id}\colon \pi_1(D_i) \to \mathrm{PSL}(2,\mathbb{C})$ is the trivial representation. Since $f|_{\ell_i} = \phi_i|_{\partial D_i}$, we can identify ℓ_i and ∂D_i. Thus we have extended C on F to the component of $\hat{F} \setminus F$ bounded by ℓ_i. By applying this extension to each ℓ_i $(i = 1, 2, \ldots, n)$, we obtain an almost good projective structure $\hat{C} = (\hat{f}, \rho_{id})$ on $\hat{F}$. If we take $p'_1, p'_2, \ldots, p'_n$ to be different points in $int(D')$, then, via $\hat{f}$, the punctures of $\sqcup_i D_i$ map to these different points in $int(D')$. Therefore, if F is a holed sphere (i.e. F has no punctures), then we can extend C to a good projective structure on a punctured sphere (i.e. no boundary components) by applying this (modified) structure extension to all boundary components of R.

3.5. **Grafting.** Grafting was initially developed as an operation that transforms a hyperbolic surface to a projective surface by inserting a flat affine cylinder along a circular loop (Maskit [13], Hejhal [5], Sullivan-Thurston [17], Kamishima-Tan [7]). Goldman used a variation of this grafting operation, which is done along a more general kind of loop, called an *admissible loop*, on a projective surface, and this operation preserves the holonomy representation ([4]). In this paper, we essentially follow the definition given by Goldman. Below, we define grafting operations in terms of minimal developing maps. In addition, we define a grafting operation along an arc, so that this operation is compatible with identifying boundary components of base surface(s).

Let F be a genus-zero surface of finite type. Then, we can set $F = \mathbb{S}^2 \setminus (D_1 \sqcup D_2 \sqcup \ldots \sqcup D_n)$, where $D_1, D_2, \ldots, D_n$ are disjoint points and disks on $\mathbb{S}^2$. Accordingly, let $D'_1, D'_2, \ldots, D'_n$ be disjoint points and disks on $\hat{\mathbb{C}}$ homeomorphic to $D_1, D_2, \ldots, D_n$, respectively. Then choose a homeomorphism $\phi\colon \mathbb{S}^2 \to \hat{\mathbb{C}}$, taking D_i to D'_i for each $i \in \{1, 2, \ldots, n\}$. Let $R := \hat{\mathbb{C}} \setminus (D'_1 \sqcup D'_2 \sqcup \ldots \sqcup D'_n)$. Let $f\colon F \to R$ be the homeomorphism obtained by restricting ϕ to F, and $\rho_{id}\colon \pi_1(F) \to \mathrm{PSL}(2,\mathbb{C})$ be the trivial representation. Then $C = (f, \rho_{id})$ be the basic projective structure on F isomorphic to the canonical projective structure on R. Let α be an arc on F connecting D_i and D_j with $i \neq j$. Then $f(\alpha) =: \beta$ is an arc on R connecting D'_i and D'_j. Let B be the canonical projective structure on $\hat{\mathbb{C}} \setminus \{D'_i, D'_j\}$. Then β is an arc on B connecting D'_i and D'_j.

Then we can transform C to another good projective structure $Gr_\alpha(F)$ on F; we cut F and B along α and β, respectively, and combine $F \setminus \alpha$ and $B \setminus \beta$ together by glueing them along their boundary arcs in an alternating fashion, which will be more precisely described in the following: First, we pick a (small regular) neighborhood N_α of α in C and a neighborhood N_β of β in B so that N_α is isomorphic to N_β via

f. Let α_1, α_2 be the boundary arcs of $C \setminus \alpha$ corresponding to α, and let β_1, β_2 be the respective boundary arcs of $B \setminus \beta$ corresponding to β; that is, for each $i = 1, 2$, the component of $N_\alpha \setminus \alpha$ bounded by α_i is isomorphic to the component of $N_\beta \setminus \beta$ bounded by β_i (see Figure 1). Then we can glue $C \setminus \alpha$ and $B \setminus \beta$ together by identifying α_1 and β_2 and identifying α_2 and β_1 using the identification of α and β via f. Thus we have obtained a surface homeomorphic to F enjoying a new good projective structure fully supported on R. This operation is called the **grafting (operation)** on C along α, and this new structure is denoted by $Gr_\alpha(C)$. It is easy to show that $Gr_\alpha(C)$ does *not* change under the isotopy of α on F. For a boundary component ℓ of F, if ℓ contains an end point of α, the restriction of $dev(Gr_\alpha(C))$ to ℓ is a covering map of degree 2 onto its image, and, otherwise, is a homeomorphism onto its image. Analogously, for a puncture p of F, if p is an end point of α, $dev(Gr_\alpha(C))$ extends continuously to p so that p is a branched point of degree 2, and, otherwise, it extends homeomorphically to p. If there is a multiarc on F of which each arc connects different D_i's, we can simultaneously graft C along this multiarc and obtain a good structure on F fully supported on R.

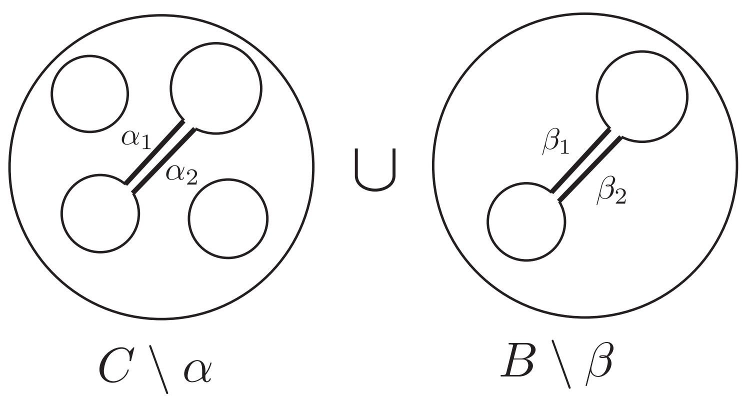

FIGURE 1. A picture for $Gr_\alpha(C)$ when F is a four-holed sphere.

Last, supposing that $C = (f, \rho)$ is a projective structure on a (general) surface F, we define grafting along a certain kind of loops on C. Let $\tilde{F}$ be the minimal cover of F associated with ρ. Let $\tilde{C}$ be the projective structure on $\tilde{F}$ obtained by lifting C. Then the holonomy of $\tilde{C}$ is trivial and $dev(\tilde{C})$ is an immersion from $\tilde{F}$ into $\hat{\mathbb{C}}$ (see §3.1). A loop ℓ on the projective surface (F, C) is called **admissible** if $\rho(\ell)$ is loxodromic and a lift $\tilde{\ell}$ of ℓ to $\tilde{F}$ injects into $\hat{\mathbb{C}}$ via f. Then $f(\tilde{\ell})$ is a (simple) arc on $\hat{\mathbb{C}}$ invariant under the infinite cyclic group $\langle \rho(\ell) \rangle$, and therefore

the end points of $f(\tilde{\ell})$ are the fixed points of $\rho(\ell)$. Denote the set of these fixed points by $Fix(\rho(\ell))$. In particular, if C is a basic structure, then a loop ℓ on (F, C) is admissible if and only if $\rho(\ell)$ is loxodromic. If ℓ is an admissible loop, $\hat{\mathbb{C}} \setminus Fix(\rho(\ell))$ is a twice-punctured sphere equipped with a canonical projective structure and $f(\tilde{\ell})$ is an arc properly embedded in $\hat{\mathbb{C}} \setminus Fix(\rho(\ell))$, connecting the punctures $Fix(\rho(\ell))$. Then we can similarly graft $\tilde{C}$ along $\tilde{\ell}$ and obtain a new projective structure $Gr_{\tilde{\ell}}(\tilde{C})$ on $\tilde{F}$ with the trivial holonomy, by identifying the boundary arcs of $\tilde{C} \setminus \tilde{\ell}$ corresponding to $\tilde{\ell}$ and the boundary arcs of $\hat{\mathbb{C}} \setminus (Fix(\rho(\ell)) \cup f(\tilde{\ell}))$ corresponding to $f(\tilde{\ell})$ in the alternating fusion. Note that $\langle \ell \rangle \cong \mathbb{Z}$ faithfully acts on $Gr_{\tilde{\ell}}(\tilde{C})$, and therefore $dev(Gr_{\tilde{\ell}}(\tilde{C}))$ is $\tilde{\rho}|_{\langle \ell \rangle}$-equivariant, where $\tilde{\rho} : \pi_1(S)/\ker(\rho) \to Im(\rho)$ is the canonical isomorphism. The **total lift** $\tilde{L}$ of ℓ to $\tilde{F}$ is the union of all lifts of ℓ to $\tilde{F}$. By grafting $\tilde{C}$ along all components of $\tilde{L}$, we obtain a Γ-invariant projective structure $Gr_{\tilde{L}}(\tilde{C})$, so that $dev(Gr_{\tilde{L}}(\tilde{C}))$ is $\tilde{\rho}$-equivariant. Then $Gr_{\tilde{L}}(\tilde{C})/\Gamma$ is a projective structure on F whose holonomy is ρ. Thus we have transformed C to $Gr_{\tilde{L}}(\tilde{C})/\Gamma$ without changing the holonomy. This operation is called a **grafting** on C along ℓ, and we denote this new structure $Gr_{\tilde{L}}(\tilde{C})/\Gamma$ by $Gr_\ell(C)$. If there is a multiloop on (F, C) consisting of admissible loops, we can graft C simultaneously along the multiloop without changing holonomy.

3.6. **Hurwitz spaces.** Let d_i $(i = 1, 2, \ldots, k)$ be integers greater than 1. Consider a pair consisting of a set of k distinct ordered points $P_i\,(i = 1, 2, \ldots, k)$ on $\hat{\mathbb{C}}$ and a rational function $\tau\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that P_i are the ramification points of τ with ramification index d_i (i.e. $\frac{d\tau(z)}{dz}$ has zero of order $d_i - 1$ at P_i), and $\tau(P_i)$ are distinct points on $\hat{\mathbb{C}}$. Let $\mathcal{R} = \mathcal{R}(d_1, d_2, \ldots, d_k)$ denote the space of all such pairs (P_i, τ). Then $\mathrm{PSL}(2, \mathbb{C})$ acts on $\mathcal{R}$ by postcomposition. The quotient space $\mathcal{R}/\mathrm{PSL}(2, \mathbb{C})$ is called a **Hurwitz space**, which we denote by $\mathcal{H} = \mathcal{H}(d_1, d_2, \ldots, d_k)$. It is well known that the map $\tau \mapsto (\tau(P_1), \tau(P_2), \ldots, \tau(P_k))$ is a covering map from $\mathcal{R}$ onto $\hat{\mathbb{C}}^k \setminus (diagonals)$, and therefore $\mathcal{R}$ is a complex manifold. Since $\mathrm{PSL}(2, \mathbb{C})$ acts on $\mathcal{R}$ holomorphically, $\mathcal{R}$ is a $\mathrm{PSL}(2, \mathbb{C})$-bundle over $\mathcal{H}$. Hence $\mathcal{H}$ is also a complex manifold. (See [2, 20].) Moreover,

Theorem 3.2 (Liu-Osserman [11])**.** *$\mathcal{H}$ is a connected manifold; hence, $\mathcal{R}$ is also a connected manifold.*

4. Decomposition of a Schottky structure into almost good holed spheres

Definition 4.1. *Let $C = (f, \rho)$ be a projective structure on a surface F. A loop ℓ on F is a* ***meridian*** *if ℓ is an essential loop on F and $\rho(\ell) = id$.*

Definition 4.2. *Let F be a surface and M be a multicurve on F. The* ***essential part*** *of M is the union of all essential loops of M. We denote the essential part of M by $\lfloor M \rfloor$.*

Lemma 4.3. *Let $(F, C = (f, \rho_{id}))$ be an almost good genus-zero surface supported on $R \subset \hat{\mathbb{C}}$. Let ℓ be an inessential loop on R or a loop contained in a component of $\hat{\mathbb{C}} \setminus R$. Then $\lfloor f^{-1}(\ell) \rfloor = \emptyset$.*

Proof. First suppose that ℓ is contained in a component D of $\hat{\mathbb{C}} \setminus R$. Since R is connected and D is planar, D must be a disk. Therefore ℓ bounds a disk E contained in D. Recall that f has the lifting property along every path p on $\hat{\mathbb{C}}$ such that p is disjoint from the punctures of R and p does *not* cross any boundary component of R. Therefore $f^{-1}(\ell)$ is a multiloop on F. Each loop m of $f^{-1}(\ell)$ bounds a component P of $f^{-1}(E)$ and P is homeomorphic to E via f. Therefore P is a disk bounded by m. Thus $f^{-1}(\ell)$ is a union of disjoint inessential loops on F and we have $\lfloor f^{-1}(\ell) \rfloor = \emptyset$.

Next suppose that ℓ is an inessential loop on R. Then ℓ bounds a disk D in R. Similarly, since the restriction of f to $f^{-1}(D)$ is a covering map onto D, $f^{-1}(D)$ is a union of disjoint disks on F. Since $f^{-1}(\ell)$ bounds $f^{-1}(D)$, we have $\lfloor f^{-1}(\ell) \rfloor = \emptyset$. $\square$

4.1. Pulling back a multiloop via a developing map.

Let Γ be a fuchsian Schottky group of rank g. (The arguments in §4 hold without the assumption that Γ is fuchsian. Even after §4, since our arguments are basically topological, this is still *not* an essential assumption. However this assumption makes it easier to formulate the arguments.) Let Ω denote the domain of discontinuity of Γ. Let $S' = \Omega/\Gamma$, which is homeomorphic to S. Let $H' = (\Omega \cup \mathbb{H}^3)/\Gamma$. Then H' is a handlebody of genus g and we have $\partial H' = S'$. Let ρ be an epimorphism from $\pi_1(S)$ onto Γ. Let $\tilde{S}$ be the minimal cover S associated with ρ, and let Ψ_Γ be the covering map from $\tilde{S}$ to S. Then Γ is the covering transformation group acting on $\tilde{S}$ (see §3.1). We see that $\tilde{S}$ is homeomorphic to $\hat{\mathbb{C}} \setminus \Lambda(\Gamma)$, where $\Lambda(\Gamma)$ is the limit set of Γ, and in particular $\tilde{S}$ is planar.

Let C be a projective structure on S with the holonomy ρ. Set $C = (f, \rho)$, where $f \colon \tilde{S} \to \hat{\mathbb{C}}$ is the (minimal) developing map of C. Recall that f is $\tilde{\rho}$-equivariant, where $\tilde{\rho} = \rho/ker(\rho)$.

Lemma 4.4. *Let $\tilde{\mu}'$ be a loop on Ω that (homeomorphically) projects to a loop on S'. Then $f^{-1}(\tilde{\mu}')$ is a multiloop on $\tilde{S}$.*

Proof. Since f has the path lifting property in Ω, $f^{-1}(\tilde{\mu}')$ is a 1-manifold properly embedded in $\tilde{S}$ (see [9]). Since f is $\tilde{\rho}$-equivariant and $\tilde{\mu}'$ projects to a loop on $\tilde{S}$, for each point $x \in f^{-1}(\tilde{\mu}')$, there exists a neighborhood U of x such that U is a 2-disk and $U \cap f^{-1}(\tilde{\mu}')$ is a single arc properly embedded in U and $f^{-1}(\tilde{\mu}') \cap \gamma U = \emptyset$ for all $\gamma \in \Gamma \setminus \{id\}$. Therefore $f^{-1}(\tilde{\mu}')/\Gamma$ is a multiloop on S. Suppose that $f^{-1}(\tilde{\mu}')$ contains a biinfinite simple curve $\tilde{\mu}$. Then, since $\tilde{\mu}$ is a lift of a loop μ of $f^{-1}(\tilde{\mu}')/\Gamma$ on S, (the homotopy class of) μ translates $\tilde{S}$ along $\tilde{\mu}$. Since $\tilde{\rho}$ is an isomorphism, $\rho(\mu)$ is loxodromic. On the other hand, since $\tilde{\mu}$ covers $\mu' \subset \Omega$ via the $\tilde{\rho}$-equivariant developing map f, $\rho(\mu)$ must be the identity element of $\mathrm{PSL}(2, \mathbb{C})$. Thus we have a contradiction. □

A loop on the boundary of a handlebody is called **meridian** if it bounds a disk properly embedded in the handlebody. Let N' be a multiloop on $S' = \partial H'$ consisting of meridian loops. Let $\tilde{N}'$ be the total lift of N' to Ω. Then $\tilde{N}'$ is a Γ-invariant multiloop on Ω. Since f is $\tilde{\rho}$-equivariant, by Lemma 4.4, $f^{-1}(\tilde{N}')$ is a Γ-invariant multiloop on $\tilde{S}$ (typically this multiloop is *not* necessarily locally finite, since there are infinitely many loops of $\tilde{N}'$ near a point of the limit set Λ of Γ). Let $\tilde{N} = \lfloor f^{-1}(\tilde{N}') \rfloor$. Let N denote the multiloop on S obtained by quotienting $\tilde{N}$ by Γ. Call N the **pullback** of N' (via f).

Proposition 4.5. *Assume that N' is a multiloop on S' satisfying:*
(I) N' consists of finitely many meridian loops and
(II) each component of $S' \setminus N'$ is a holed sphere.
Then the pullback N of N' via f satisfies:
(i) $N \neq \emptyset$,
(ii) N consists of finitely many meridian loops on S (with respect to ρ), and
(iii) if Q is a component of $S \setminus N$, then $C|Q$, the restriction of C to Q, is an almost good holed sphere supported on a component of $\Omega \setminus \tilde{N}'$.

This proposition immediately implies:

Corollary 4.6. *Let $\tilde{C} = (f, \rho_{id})$ denote the projective structure on $\tilde{S}$ obtained by lifting C. (i) Suppose that $\tilde{Q}$ is a component of $\tilde{S} \setminus \tilde{N}$. Then there is a unique component R of $\Omega \setminus \tilde{N}'$ such that $\tilde{C}|\tilde{Q}$ is an almost good holed sphere supported on R (in particular $f(\partial\tilde{Q}) \subset \partial R$). (ii) Suppose that $\tilde{Q}_1$ and $\tilde{Q}_2$ are adjacent components of $\tilde{S} \setminus \tilde{N}$, sharing a*

loop ℓ of $\tilde{N}$ as a boundary component. Then the supports of $\tilde{C}|\tilde{Q}_1$ and $\tilde{C}|\tilde{Q}_2$ are adjacent components of $\Omega \setminus \tilde{N}'$, sharing $f(\ell)$ as a boundary component.

Proof of Proposition 4.5 (ii). First we claim that, for each $x \in \tilde{S}$, there is a neighborhood U of x such that U is topologically a closed disk and $U \cap \tilde{N}$ is either the empty set or a single arc properly embedded in U. Since f is a local homeomorphism, we can take an open neighborhood $U$of x such that U is homeomorphic to a closed disk and $f|_U$ is a homeomorphism onto its image $f(U) =: U'$. Clearly, we have $\hat{\mathbb{C}} = \Lambda \sqcup \tilde{N}' \sqcup (\Omega \setminus \tilde{N}')$.
Case 1. Suppose that $f(x) \in \Lambda$. Then, by Assumption (II), we can assume that U' is a closed disk bounded by a loop of $\tilde{N}'$. Then ∂U is a component of $f^{-1}(\tilde{N}')$, and it is an inessential loop on $\tilde{S}$. Moreover, the pair $(U, f^{-1}(\tilde{N}') \cap U)$ is homeomorphic to the pair $(U', \tilde{N}' \cap U')$ via f. Then $\tilde{N}' \cap U'$ is a union of infinitely many disjoint loops that are inessential in U'. Accordingly $f^{-1}(\tilde{N}') \cap U$ is a union of disjoint loops in the disk U, and thus we have $\tilde{N} \cap U = \lfloor f^{-1}(\tilde{N}') \cap U \rfloor = \emptyset$.
Case 2. Suppose that $f(x) \in \tilde{N}'$. Then, since $\tilde{N}$ is a submanifold of $\tilde{S}$, we can take U so that U' intersects $\tilde{N}'$ only in a single arc A properly embedded in U'. Then $f^{-1}(\tilde{N}') \cap U = f^{-1}(A) \cap U$ is a single arc properly embedded in U. Therefore $\lfloor f^{-1}(\tilde{N}') \rfloor \cap U$ is either the empty set or $f^{-1}(A) \cap U$.
Case 3. Last suppose that $f(x) \in \Omega \setminus \tilde{N}'$. Then we can take U so that $f(U)$ is disjoint from $\tilde{N}'$. Then $f^{-1}(\tilde{N}') \cap U = \emptyset$ and therefore $\lfloor f^{-1}(\tilde{N}') \rfloor \cap U = \emptyset$.

The surface S is closed and the covering map Ψ_Γ is a local homeomorphism from $(\tilde{S}, \tilde{N})$ to (S, N). Therefore, by the claim above, there is a finite cover $\{U_i\}$ of S such that, for each i, U_i is a closed disk and $U_i \cap N$ is either the empty set or an arc properly embedded in U_i. Thus N is a multiloop on S containing only finitely many loops.

Next, we show that N consists only of meridian loops. By the definition of $\tilde{N}$, it is clear that $\tilde{N}$ consists of essential loops on $\tilde{S}$. Let ℓ be a loop of N. Then ℓ lifts to a loop of $\tilde{N}$ on $\tilde{S}$. Therefore, by the definition of $\tilde{S}$, $\rho(\ell) = id$. Thus ℓ is meridian. □

Lemma 4.7. *Assume (I) and (II) in Proposition 4.5. Then, for every component Q of $S \setminus N$, the restriction of ρ to $\pi_1(Q)$ is the trivial representation.*

Proof. For $\alpha \in \pi_1(Q)$, let $\gamma = \rho(\alpha) \in \Gamma$. We regard α also as an oriented closed curve on Q representing α. Suppose that $\gamma \neq id$. Then

γ is a loxodromic element. Therefore α lifts a bi-infinite simple curve $\tilde{\alpha}$ on $\tilde{S}$ invariant under the action of $\langle\alpha\rangle$, the infinite cyclic group generated by α. Let $p_1, p_2 \in \hat{\mathbb{C}}$ be the fixed points of γ. Then $f|_{\tilde{\alpha}}$ is a $\rho|_{\langle\alpha\rangle}$-equivariant curve connecting p_1 and p_2. By Assumption (II), there is a loop μ' of $\tilde{N}'$ separating p_1 and p_2. By a small isotopy of α on Q, if necessary, we can assume that the curve $f|_{\tilde{\alpha}}$ does *not* intersect p_1 and p_2 and transversally intersects μ'. Since $f|_{\tilde{\alpha}}$ is $\rho|_{\langle\alpha\rangle}$-equivariant and p_1, p_2 are contained in the different components of $\hat{\mathbb{C}} \setminus \mu'$, $f|_{\tilde{\alpha}}$ intersects μ' an odd number of times. Therefore $\tilde{\alpha}$ transversally intersects $f^{-1}(\mu')$ an odd number of times.

By Lemma 4.4, $f^{-1}(\mu')$ is a multiloop on $\tilde{S}$. Furthermore, by Proposition 4.5 (ii), each component of $f^{-1}(\mu')$ is either a loop of $\tilde{N}$ or an inessential loop on $\tilde{S}$. Note that $\tilde{\alpha}$ is a $\langle\alpha\rangle$-invariant curve properly immersed in $\tilde{S}$ and it is, in particular, unbounded. Therefore, each inessential loop of $f^{-1}(\mu')$ intersects $\tilde{\alpha}$ an even number of times. Since $\tilde{\alpha}$ intersects $f^{-1}(\mu')$ an odd number of times, $\tilde{\alpha}$ must intersect at least one essential loop of $f^{-1}(\mu')$. Thus $\tilde{\alpha}$ intersects $\tilde{N}$, and therefore α transversally intersects N. This contradicts the assumption that α is contained in Q. Hence $\rho(\alpha) = id$ for all $\alpha \in \pi_1(Q)$. $\square$

Proposition 4.5 (i) immediately follows from

Lemma 4.8. *Every component of $S \setminus N$ has genus zero.*

Proof. Let Q be a component of $S \setminus N$. By Lemma 4.7, $\pi_1(Q) \subset \ker \rho$. Thus, by the definition of $\tilde{S}$, Q homeomorphically lifts to a component $\tilde{Q}$ of $\tilde{S} \setminus \tilde{N}$. Since $\tilde{S}$ is planar, Q has genus 0. $\square$

Proof Proposition 4.5 (iii). Let Q be a component of $S \setminus N$. By Lemma 4.8, Q is a holed sphere. Since Q is bounded by essential loops on S, Q has at least two boundary components. By Lemma 4.7, the holonomy of $C|Q$ is trivial. Therefore we can homeomorphically lift Q to a subsurface $\tilde{Q}$ of $\tilde{S}$. Then we can set $dev(C|Q) = f|_{\tilde{Q}} \colon \tilde{Q} \cong Q \to \hat{\mathbb{C}}$.

We show that there is a component R of $\Omega \setminus \tilde{N}'$ such that $\partial\tilde{Q}$ covers ∂R via f (note that this covering map is *not* necessarily onto). Let ℓ be a boundary component of $\tilde{Q}$. Then $f(\ell)$ is a loop of $\tilde{N}'$. Take a small neighborhood N_ℓ of ℓ in $\tilde{Q}$ so that there is a component R of $\Omega \setminus \tilde{N}'$ satisfying $f(N_\ell) \subset R$. We claim that, if m is another boundary component of Q, then $f(m)$ is also a boundary component of R. Let $\alpha \colon [0,1] \to S$ be an arc properly embedded in $\tilde{Q}$ that connects ℓ to m. Suppose that $f(m)$ is *not* a boundary component of R. Then $f(m)$ is contained in the interior of a component D of $\hat{\mathbb{C}} \setminus R$. Then D is a disk

bounded by either $f(\ell)$ or a boundary component of R different from $f(\ell)$.

Case 1. Suppose that D is bounded by a boundary component n' of R different from $f(\ell)$. Then $f(\ell)$ and $f(m)$ are contained in different components of $\hat{\mathbb{C}} \setminus n'$. We can assume that the curve $f \circ \alpha$ is transversal to n' by a small isotopy of α, if necessary. Since $f \circ \alpha$ is a (*not* necessarily simple) curve on $\hat{\mathbb{C}}$ connecting $f(\ell)$ to $f(m)$, $f \circ \alpha$ intersects n' an odd number of times. Therefore α transversally intersects $f^{-1}(n')$ an odd number of times. By Lemma 4.4, $f^{-1}(n')$ is a multiloop on $\tilde{S}$. Let n be an inessential loop of $f^{-1}(n')$ that intersects α. Then n bounds a closed disk E in $\tilde{S}$. Since ℓ and m are essential loops on $\tilde{S}$ and disjoint from n, they are contained in $\tilde{S} \setminus E$. Therefore α intersects n an even number of times. Since α transversally intersects $f^{-1}(n')$ an odd number of times, α must intersect an essential loop of $f^{-1}(n')$. Therefore α transversally intersects $\tilde{N}$. This contradicts the assumption that α is in $\tilde{Q}$.

Case 2. Suppose that $f(\ell)$ bounds D. Then, for sufficiently small $\epsilon > 0$, $\alpha((0, \epsilon))$ is contained in the interior of R. Then the point $f \circ \alpha(\epsilon)$ and the loop $f(m)$ are contained in different components of $\hat{\mathbb{C}} \setminus f(\ell)$. Similarly, we can assume that the curve $f \circ \alpha$ transversally intersects $f(\ell)$. Then $f \circ \alpha$ transversally intersects $f(\ell)$ an odd number of times (note that $f \circ \alpha(0)$ is *not* a transversal intersection point). As in Case 1, the analysis of the intersection between $f^{-1}(f(\ell))$ and α induces the contradiction that α transversally intersects $\tilde{N}$.

By Cases 1 and 2, we see that $\partial \tilde{Q}$ covers ∂R via f.

Last, we show that $f(\partial \tilde{Q})$ consists of at least two boundary components of R. Suppose that $f(\partial \tilde{Q})$ is a single boundary component ℓ' of R. Then $f|_{\tilde{Q}} \colon \tilde{Q} \to \hat{\mathbb{C}}$ has the path lifting property on $\hat{\mathbb{C}} \setminus \ell'$. Since $f^{-1}(\ell')$ is a union of disjoint loops on $\tilde{S}$, $f^{-1}(\ell') \cap \tilde{Q}$ is a union of disjoint loops on $\tilde{Q}$. Let P be a component of $\tilde{Q} \setminus f^{-1}(\ell')$ that shares a boundary component m with $\tilde{Q}$. Then $f|_P$ is a covering map from P onto a component of $\hat{\mathbb{C}} \setminus \ell'$, which is a disk. Therefore P is also a disk. This contradicts the assumption that m is an essential loop on $\tilde{S}$. $\square$

5. Schottky structures and handlebodies with cellular structures.

(Many arguments in this section are analogous to ones in [15].) We carry over our notation from §4. Let Ω_0 be a standard fundamental domain of the Γ-action on Ω, i.e. Ω_0 is a connected region in $\hat{\mathbb{C}}$ bounded by $2g$ disjoint round circles orthogonal to the equator $\mathbb{R} \cup \{\infty\}$. We assume that Ω_0 is a closed region. The boundary components of Ω_0

are paired up and identified by generators $\gamma_1, \gamma_2, \ldots, \gamma_g$ of Γ. Then $\partial\Omega_0$ is a multiloop in Ω, and its Γ-orbit $\Gamma(\partial\Omega_0) =: \tilde{L}'$ is a Γ-invariant multiloop splitting Ω into connected fundamental domains of Γ. Let $L' = \tilde{L}'/\Gamma$. Recall that $S' = \Omega/\Gamma$ is the boundary surface of the genus-g handlebody $H' = (\mathbb{H}^3 \cup \Omega)/\Gamma$. Then L' is a multiloop on S' consisting of g meridian loops of H', such that $S' \setminus L'$ is a $2g$-holed sphere. Let L be the pullback of L' via f, which is a multiloop on S (see §4.1). Thus we can apply Proposition 4.5 to $N = L$. Let $\tilde{L} = \lfloor f^{-1}(\tilde{L}') \rfloor$. Then $\tilde{L}$ is the total lift of L to $\tilde{S}$ by the definition of L.

5.1. **Cellular structures on handlebodies.** For a subset $X \subset \mathbb{H}^3 \cup \Omega$, let $Conv(X)$ denote the convex hull of X in $\mathbb{H}^3 \cup \Omega$ ($\subset \overline{\mathbb{H}^3} = \mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3$). Then, for each loop ℓ of $\tilde{L}'$, $Conv(\ell)$ is a copy of $\overline{\mathbb{H}^2}$. Let $\tilde{\Delta}' = \sqcup Conv(\ell)$, where the union runs over all loops ℓ of $\tilde{L}'$. Then $\tilde{\Delta}'$ is a multidisk in $\mathbb{H}^3 \cup \Omega$, and each component of $(\mathbb{H}^3 \cup \Omega) \setminus \tilde{\Delta}'$ is a fundamental domain of the Γ-action on $\mathbb{H}^3 \cup \Omega$. Let $\Delta' = \tilde{\Delta}'/\Gamma$. Then Δ' is a union of g disjoint copies of $\overline{\mathbb{H}^2}$ bounded by L', and Δ' splits H' into a 3-disk. Thus, we can regard the pair (H', Δ') as a handlebody with a cellular structure consisting of g 2-cells, the disks of Δ', and one 3-cell, $H' \setminus \Delta'$.

By Proposition 4.5 (iii), each component of $S \setminus L$ is a sphere with at least 2 holes. Letting H be a genus g handlebody, we can identify S with ∂H so that each loop of L is a meridian loop. Let Δ be the multidisk bounded by L and embedded properly in H. Then Δ splits H into finitely many 3-disks. Thus we can regard (H, Δ) as a handlebody with a cellular structure whose 2-cells are the disks of Δ and 3-cells are the components of $H \setminus \Delta$. Let $\tilde{H}$ denote the universal cover of H, so that $\partial\tilde{H} = \tilde{S}$. Let $\tilde{\Delta}$ denote the total lift of Δ to $\tilde{H}$, which is a Γ-invariant multidisk bounded by $\tilde{L}$.

In this section we prove:

Proposition 5.1. *There exists an embedding $\epsilon\colon (H, \Delta) \to (H', \Delta')$ with the following properties:*
(i) For each $d = 2, 3$, ϵ embeds each d-cell of (H, Δ) into a d-cell of (H', Δ'),
(ii) $H' \setminus int(Im(\epsilon))$ is homeomorphic to $S \times [0, 1]$, and
(iii) if ℓ is a loop of $\tilde{L}$, then $\tilde{\epsilon}(\ell) \subset Conv(f(\ell)) \cong \mathbb{H}^2$, where $\tilde{\epsilon}$ is the lift of ϵ to an embedding of $(\tilde{H}, \tilde{\Delta})$ into $(\mathbb{H}^3 \cup \Omega, \tilde{\Delta}')$.

Remarks: In (ii), $S \times \{0, 1\}$ corresponds to $\partial H' \sqcup \epsilon(\partial H)$. In (iii), the existence of the lift $\tilde{\epsilon}$ is guaranteed by (ii) (since ϵ is π_1-injective). It turns out that Proposition 5.1 is equivalent to that with (iii) replaced

by: (iii') ρ coincides with the homomorphism $(\epsilon|_S)^\star\colon \pi_1(S) \to \pi_1(H') = \Gamma$, induced by the embedding $\epsilon|_{\partial H}\colon S \to H'$. (The outline of the proof of this equivalence: observe that f is ρ-equivariant and (iii') is equivalent to saying that $\tilde{\epsilon}$ is ρ-equivariant; let $\epsilon^*\colon (\tilde{H}, \tilde{\Delta})^* \to (\tilde{H}', \tilde{\Delta}')^*$ be the graph map induced by ϵ (see §5.2, 5.3), and, assuming (iii'), analyze loops in $(\tilde{H}, \tilde{\Delta})^*$ that ϵ^* embeds into $(\tilde{H}', \Delta')^*$.)

In addition, by Proposition 5.1 (i), (iii), and Proposition 4.5 (iii), we immediately obtaion

Corollary 5.2. *If R is a component of $\tilde{S} \setminus \tilde{L}$, then R is properly embedded into $Conv(Supp_f(R))$ by $\tilde{\epsilon}$.*

5.2. **Dual graphs of cellular handlebodies.** Let (M, Δ_M) be a pair consisting of a 3-manifold with boundary and a union of isolated 2-disks D_i (with $i \in I$) properly embedded in M. Pick pairwise disjoint regular neighborhoods N_i of D_i ($i \in I$), such that D_i are homeomorphic to $D_i \times [-1, 1]$ and that $N_i \cap \partial M$ are homeomorphic to $\partial D_i \times [-1, 1]$. For each $i \in I$ and $x \in [-1, 1]$, collapse each $D_i \times \{x\}$ to a single point and also each component of $M \setminus \sqcup_i N_i$ to a single point. Then the resulting quotient space is a graph whose edges bijectively correspond to N_i ($i \in I$) and vertices to the components of $M \setminus \sqcup_i N_i$. This graph is called the **dual graph** of (M, Δ_M) and denoted by $(M, \Delta_M)^*$. Similarly, if X is an edge or vertex of $(M, \Delta_M)^*$ or a cell of (M, Δ_M) , we let X^* denote its appropriate dual object under the duality between (M, Δ_M) and $(M, \Delta_M)^*$. We see that the dual graph $(M, \Delta_M)^* =: G_M$ can be embedded in (M, Δ_M), realizing the duality: Each vertex of G_M is in the corresponding component of $M \setminus \sqcup_i N_i$ and each edge of G_M transversally intersects Δ_M in a single point contained in its dual disk D_i (see Figure 2).

Now we, in addition, assume that M is a handlebody of genus g and Δ_M is a union of finitely many disjoint meridian disks in M such that Δ_M splits M into 3-disks. In particular, (H, Δ) and (H', Δ') in §5.1 satisfy the assumptions for (M, Δ_M) Clearly, G_M is a finite connected graph and, since a meridian disk is *not* boundary parallel, every vertex of G_M has degree at least 2. Since Δ_M splits M into 3-disks, we can canonically choose the above embedding of G_M into (M, Δ_M) so that it also satisfies $M \setminus G_M \cong \partial M \times (0, 1]$, where ∂M on the left is identified with $\partial M \times \{1\}$ on the right. Then G_M is a deformation retract of M, and, therefore, $\pi_1(G_M)$ is isomorphic to the rank-g free group. Let $\tilde{M}$ denote the universal cover of M, and let $\tilde{\Delta}_M$ be the total lift of Δ_M to $\tilde{M}$. Let $\tilde{G}_M$ be the universal cover of G_M. Then $\tilde{G}_M$ is the dual graph of $(\tilde{M}, \tilde{\Delta}_M)$. Since $\pi_1(M)$ acts on $\tilde{G}_M$ and $(\tilde{M}, \tilde{\Delta}_M)$, preserving their

cellular structures, for every $\gamma \in \pi_1(M)$ and every cell x of $\tilde{G}_M$ and $(\tilde{M}, \tilde{\Delta}_M)$, we have $(\gamma x)^* = \gamma x^*$.

Conversely, given a finite connected graph K, we can easily construct a pair of a handlebody H_K and the union of disjoint meridian disks Δ_K of H_K such that each component of $H_K \setminus \Delta_k$ is a 3-disk and $K = (H_K, \Delta_K)^*$: Indeed, we can take a small regular neighborhood H_K of K (by embedding K linearly into $\mathbb{R}^3$), which is a handlebody. Then, for each edge of K, pick a meridian disk in H_K that intersects K once in the middle point of the edge. Thus Δ_K is realized as a union of such meridian disks.

Now let G be the dual graph of (H, Δ) and let G' be the dual graph of (H', Δ'). Then the dual graph G' is a bouquet of g circles consisting of g edges and one vertex (see Figure 2).

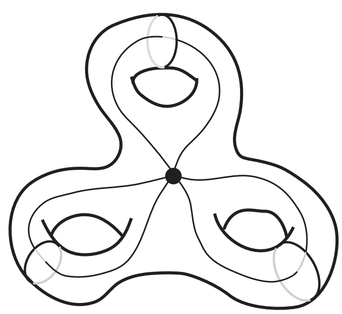

FIGURE 2. G' embedded in (H', D') in the case of $g = 3$.

5.3. **Graph homomorphisms.** A **graph homomorphism** is a simplicial map between graphs that maps each edge onto an edge (and each vertex to a vertex). We shall construct a graph homomorphism $\kappa\colon G \to G'$ naturally induced by f. Since $\tilde{S} = \partial\tilde{H}$ and Γ acts on both $\tilde{S}$ and $\tilde{H}$ in compatible ways, we can canonically identify $\pi_1(H)$ with $\pi_1(S)/\ker\rho = \Gamma$. Then $\tilde{\rho}\colon \pi_1(S)/\ker(\rho) \to \Gamma$ can also be regarded as an isomorphism from $\pi_1(H)$ to $\pi_1(H')$ and, by duality, from $\pi_1(G)$ to $\pi_1(G')$. Letting $\tilde{G}$ and $\tilde{G}'$ be the universal covers of G and G', respectively, we shall construct a $\tilde{\rho}$-equivariant graph homomorphism $\tilde{\kappa}\colon \tilde{G} \to \tilde{G}'$. First we define $\tilde{\kappa}$ on the vertices of $\tilde{G}$: Let v be a vertex of $\tilde{G}$. Then v^* is a component of $\tilde{H} \setminus \tilde{\Delta}$, and $v^* \cap \tilde{S}$ is a component of $\tilde{S} \setminus \tilde{L}$. By Corollary 4.6 (i), the almost good holed-sphere $\tilde{C}(v^* \cap \tilde{S})$ is supported on $\gamma\Omega_0$ for a unique $\gamma \in \Gamma$. Then $Conv(\gamma\Omega_0)$ is a component of $(\mathbb{H}^3 \cup \Omega) \setminus \tilde{\Delta}'$. Define $\tilde{\kappa}(v)$ to be $(Conv(\gamma\Omega_0))^*$, a vertex of $\tilde{G}'$.

Lemma 5.3. *(i) $\tilde{\kappa}$ is ρ-equivariant. (ii) Let v_1 and v_2 be the adjacent vertices of $\tilde{G}$. Then $\tilde{\kappa}(v_1)$ and $\tilde{\kappa}(v_2)$ are also adjacent vertices of $\tilde{G}'$.*

Proof. (i). As above, let v be a vertex of $\tilde{G}$ and set $Supp_f(v^* \cap \tilde{S}) = \gamma\Omega_0$ with a unique $\gamma \in \Gamma$. Recall that, for all $\omega \in \Gamma$, we have $(\omega \cdot v)^* = \omega \cdot v^*$. Since f is ρ-equivariant,

$$Supp_f((\omega \cdot v)^* \cap \tilde{S}) = Supp_f(\omega \cdot (v^* \cap \tilde{S})) = \omega \cdot Supp_f(v^* \cap \tilde{S}) = \omega \cdot \gamma\Omega_0.$$

We also have $\omega \cdot Conv(\gamma\Omega_0) = Conv(\omega \cdot \gamma\Omega_0)$ and then $\omega \cdot (Conv(\gamma\Omega_0))^* = (Conv(\omega \cdot \gamma\Omega_0))^*$. Thus $\omega \cdot \tilde{\kappa}(v) = (Conv(\omega \cdot \Omega_0))^* = \tilde{\kappa}(\omega \cdot v)$.

(ii). Since v_1 and v_2 are adjacent, there is an edge e of $\tilde{G}$ connecting v_1 and v_2. Since e^* is a disk of $\tilde{\Delta}$, $e^* \cap \tilde{S}$ is a loop of $\tilde{L}$, and then $v_1^* \cap \tilde{S}$ and $v_2^* \cap \tilde{S}$ are adjacent components of $\tilde{S} \setminus \tilde{L}$, sharing $e^* \cap \tilde{S}$ as a boundary component. By Corollary 4.6 (ii), $Supp_f(v_1^* \cap \tilde{S})$ and $Supp_f(v_2^* \cap \tilde{S})$ are adjacent components of $\Omega \setminus \tilde{L}'$, sharing $f(e^* \cap \tilde{S})$ as a boundary component. Therefore $\tilde{\kappa}(v_1)$ and $\tilde{\kappa}(v_2)$ are adjacent vertices of $\tilde{G}'$. □

By Lemma 5.3 (ii), $\tilde{\kappa}$ uniquely extends to the graph homomorphism defined on the entire graph $\tilde{G}$: For each oriented edge $[v_1, v_2]$ with adjacent vertices v_1 and v_2 of $\tilde{G}$, define $\tilde{\kappa}([v_1, v_2])$ to be $[\tilde{\kappa}(v_1), \tilde{\kappa}(v_2)]$. By Lemma 5.3 (i), $\tilde{\kappa}$ is ρ-equivariant and therefore, quotienting $\tilde{\kappa}$ by Γ, we obtain a graph homomorphism $\kappa \colon G \to G'$.

5.4. **Labeling.** Recall that $\pi_1(H') = \pi_1(G') = \Gamma$. Let e be an oriented edge of G'. Then e can be regarded as a simple closed curve on G' and its homotopy class is a unique element of $\{\gamma_1^{\pm}, \gamma_2^{\pm}, \ldots, \gamma_g^{\pm}\}$. We call this element the **label** of e and denote it by $label(e)$.

We let Ψ_Γ denote the covering map induced by the Γ-action on a space X, where $X = \tilde{S}, \Omega, \tilde{G}$, or $\tilde{G}'$, whichever is appropriate in the context. We shall show that the labels on the oriented edges of G' by elements of $\{\gamma_1^{\pm}, \gamma_2^{\pm}, \ldots, \gamma_g^{\pm}\}$ induce the unique labels on the oriented edges of $G, \tilde{G}, \tilde{G}'$ by the same elements so that the labels are preserved under the graph homomorphisms κ, $\tilde{\kappa}$, $\Psi_\Gamma \colon \tilde{G} \to G$ and $\Psi_\Gamma \colon \tilde{G}' \to G'$. First, for each oriented edge e of $\tilde{G}'$, define $label(e)$ to be $label(\Psi_\Gamma(e))$. Then this labeling on the edges of $\tilde{G}'$ is Γ-invariant. Next for each oriented edge e of $\tilde{G}$, define $label(e)$ to be $label(\tilde{\kappa}(e))$. Since $\tilde{\kappa}$ is ρ-equivariant, this labeling on the edges of $\tilde{G}$ is Γ-invariant. For each edge e of G, define $label(e)$ to be $label(\tilde{e})$, where $\tilde{e}$ is a lift of e to $\tilde{G}$. Since the labeling on the edges of $\tilde{G}$ is Γ-invariant, $label(e)$ does *not* depend on the choice of the lift $\tilde{e}$. Then $label(e) = label(\tilde{e}) =$

$label(\tilde{\kappa}(\tilde{e})) = label(\Psi_\Gamma \circ \tilde{\kappa}(\tilde{e})) = label(\kappa(e))$. Therefore κ also preserves the labels.

5.5. **Folding maps.** (See [16].) Let K be a graph (whose edges are) labeled with the elements of $\{\gamma_1^\pm, \gamma_2^\pm, \ldots, \gamma_g^\pm\}$. Assume that there are two different oriented eges $e_1 = [u, v_1]$ and $e_2 = [u, v_2]$ of K with $v_1 \neq v_2$, sharing a vertex u, and $label(e_1) = label(e_2)$. Then we can naturally identify e_1 and e_2, yielding a new labeled graph K' (see Figure 3). This operation is called a **folding (operation)** and the graph homomorphism $\mu\colon K \to K'$ realizing this folding operation is called a **folding map**. Note that a folding operation decreases the number of edges in K by 1. Since K and K' are homotopy equivalent, μ induces an isomorphism $\mu^\star\colon \pi_1(K) \to \pi_1(K')$. Using the covering theory for graphs ([16, §3.3]), we see that μ lifts to a $\mu^\star$-equivariant graph homomorphism $\tilde{\mu}$ from the universal cover of K to that of K'.

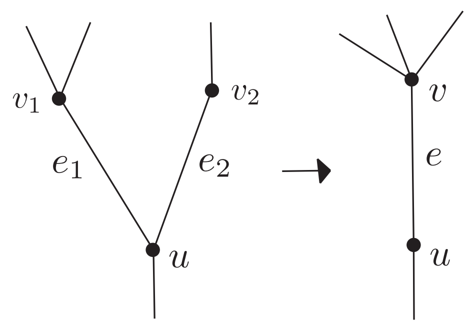

FIGURE 3. Folding.

Recall the graph homomorphism $\kappa\colon G \to G'$ from §5.3.

Lemma 5.4. *There are a sequence of folding maps,* $G = G_0 \xrightarrow{\mu_1} G_1 \xrightarrow{\mu_2} \ldots \xrightarrow{\mu_n} G_n$*, and a graph isomorphism* $\iota\colon G_n \to G'$ *such that* $\kappa = \iota \circ \mu_n \circ \mu_{n-1} \circ \ldots \circ \mu_1$.

Proof. By [16, §3.3], $\kappa\colon G \to G'$ is the composition of a sequence of folding maps $G_0 \xrightarrow{\mu_1} G_1 \xrightarrow{\mu_2} G_2 \xrightarrow{\mu_3} \ldots \xrightarrow{\mu_n} G_n$ and an immersion $\iota\colon G_n \to G'$. Since $\kappa, \mu_1, \mu_2, \ldots, \mu_n$ are homotopy equivalences, ι is also a homotopy equivalence. Since G and G' are *minimal*, i.e. they have no vertex of degree one, ι is an isomorphism. $\square$

5.6. **Proof of Proposition 5.1.** Set $\kappa = \iota \circ \mu_n \circ \mu_{n-1} \circ \ldots \circ \mu_1 \colon G = G_0 \to G'$ as in Lemma 5.3. Then, for $i \in \{1, 2, \ldots, n\}$, the folding map $\mu_i \colon G_{i-1} \to G_i$ lifts to a graph homomorphism $\tilde{\mu}_i \colon \tilde{G}_{i-1} \to \tilde{G}_i$ equivariant under the isomorphism $\mu_i^\star \colon \pi_1(G_{i-1}) \to \pi_1(G_i)$, and similarly $\iota \colon G_n \to G'$ to $\tilde{\iota} \colon \tilde{G}_n \to \tilde{G}'$ equivariant under $\iota^\star \colon \pi_1(G_n) \to \pi_1(G')$. Let $\kappa_0 = \kappa$ and $\tilde{\kappa}_0 = \tilde{\kappa}$. For each $i \in \{1, 2, \ldots, n\}$, let $\kappa_i = \iota \circ \mu_n \circ \mu_{n-1} \circ \ldots \circ \mu_{i+1} \colon G_i \to G'$ and let $\tilde{\kappa}_i = \tilde{\iota} \circ \tilde{\mu}_n \circ \tilde{\mu}_{n-1} \circ \ldots \circ \tilde{\mu}_{i+1} \colon \tilde{G}_i \to \tilde{G}'$. Then $\tilde{\kappa}_i$ is $\kappa_i^\star$-equivariant and $\tilde{\kappa}_i \circ \tilde{\mu}_i = \tilde{\kappa}_{i-1}$ for each $i \in \{0, 1, \ldots, n\}$.

Set $H_0 = H$ and $\Delta_0 = \Delta$. For each $i \in \{1, 2, \ldots, n\}$, set $(H_i, \Delta_i) = G_i^*$, where H_i is a genus g-handlebody and Δ_i is a union of disjoint meridian disks in H_i (see §5.2). For each $i \in \{0, 1, \ldots, n\}$, let $L_i = \partial \Delta_i$, which is the multiloop on ∂H_i bounding Δ_i. Let $\tilde{H}_i$ denote the universal cover of H_i. Let $\tilde{\Delta}_i$ and $\tilde{L}_i$ denote the total lifts of Δ_i and L_i to $\tilde{H}_i$, respectively, so that $\partial \tilde{\Delta}_i = \tilde{L}_i$. (Note that $\tilde{L}_0 = \tilde{L}$, $\tilde{\Delta}_0 = \tilde{\Delta}$, and $\tilde{H}_0 = \tilde{H}$.)

For a multiloop M, let $[M]$ denote the set of all loops of M. We shall define a $\kappa_i^\star$-equivariant map $f_i \colon [\tilde{L}_i] \to [\tilde{L}']$ for each $i \in \{0, 1, \ldots, n\}$. For each loop ℓ of $\tilde{L}_i$, let D_ℓ denote the disk of $\tilde{\Delta}_i$ bounded by ℓ. Then D_ℓ^* is an edge of $\tilde{G}_i$. Then $\tilde{\kappa}_i(D_\ell^*)$ is an edge of $\tilde{G}'$, and $(\tilde{\kappa}_i(D_\ell^*))^*$ is a disk of $\tilde{\Delta}'$. Define $f_i \colon [\tilde{L}_i] \to [\tilde{L}']$ by $f_i(\ell) = \partial(\tilde{\kappa}_i(D_\ell^*))^*$, the loop of $\tilde{L}'$ bounding $(\tilde{\kappa}_i(D_\ell^*))^*$. Note that $f_0 \colon [\tilde{L}_0] = [\tilde{L}] \to [\tilde{L}']$ can also be defined as the correspondence given by the covering map $f|_{\tilde{L}} \colon \tilde{L} \to \tilde{L}'$. Since $\tilde{\kappa}_i$ is $\kappa_i^\star$-equivariant, f_i is also $\kappa_i^\star$-equivariant. Since $\tilde{\mu}_i$ is $\tilde{\mu}_i^\star$-equivariant, $\tilde{\mu}_i$ similarly induces a $\mu_i^\star$-equivariant map $h_i \colon [\tilde{L}_{i-1}] \to [\tilde{L}_i]$. In addition, since $\tilde{\kappa}_{i-1} = \tilde{\kappa}_i \circ \tilde{\mu}_i$, we have $f_{i-1} = f_i \circ h_i$.

The following proposition induces Proposition 5.1 when $i = 0$.

Proposition 5.5. *For each $i \in \{0, 1, 2, \ldots, n\}$, there exists an embedding $\epsilon_i \colon (H_i, \Delta_i) \to (H', \Delta')$ such that*
(i) for $d = 2, 3$, this embedding ϵ_i takes each d-cell of (H_i, Δ_i) into a d-cell of (H', Δ'),
(ii) $H' \setminus int(Im(\epsilon_i))$ is homeomorphic to $S \times [0, 1]$, and
(iii) if ℓ is a loop of $\tilde{L}_i$, then $\tilde{\epsilon}_i(\ell) \subset Conv(f_i(\ell)) \cong \mathbb{H}^2$, where $\tilde{\epsilon}_i$ is the lift of ϵ_i to an embedding of $(\tilde{H}_i, \tilde{\Delta}_i)$ into $(\mathbb{H}^3 \cup \Omega, \tilde{\Delta}')$.

Proof. First we shall construct an embedding $\epsilon_n \colon (H_n, \Delta_n) \to (H', \Delta')$ satisfying (i), (ii), (iii). Recall that G' can be canonically embedded in (H', Δ'), realizing the duality between G' and (H', Δ'), such that $H' \setminus G' \cong S \times (0, 1]$. Take a (small) closed regular neighborhood N of G' in (H', Δ') so that $(H', \Delta') = (H_n, \Delta_n)$ is naturally isomorphic to $(N, N \cap \Delta')$ and $H' \setminus int(N) \cong S \times [0, 1]$. Let $\epsilon_n \colon (H_n, \Delta_n) \to (N, N \cap \Delta') \subset$

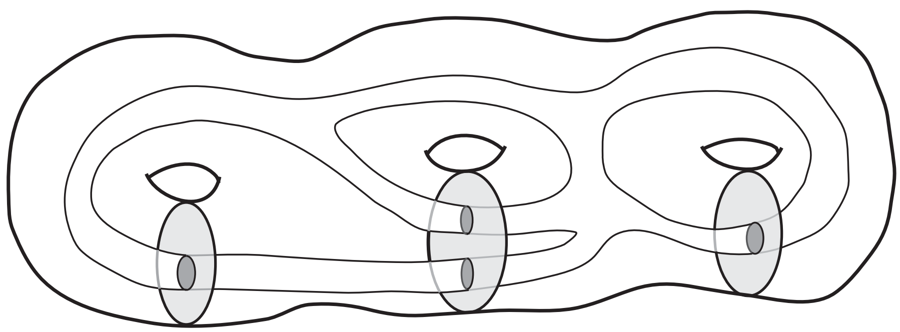

FIGURE 4. A basic example of ϵ_i.

(H', Δ') denote this isomorphism. Then ϵ_n clearly satisfies (i) and (ii). By (ii), ϵ_n lifts to a $\kappa_n^\star$-equivariant isomorphism $\tilde{\epsilon}_n \colon (\tilde{H}_n, \tilde{\Delta}_n) \to (\tilde{N}, \tilde{N} \cap \tilde{\Delta}') \subset (\mathbb{H}^3 \cup \Omega, \tilde{\Delta}')$, where $\tilde{N}$ is the total lift of N to $\mathbb{H}^3 \cup \Omega$. Then we can embed $\tilde{G}_n$ into $(\tilde{H}_n, \tilde{\Delta}_n)$ and $\tilde{G}'$ into $(\mathbb{H}^3 \cup \Omega, \tilde{\Delta}')$, realizing their dualities, such that those embeddings are Γ-invariant and the isomorphism $\tilde{\kappa}_n \colon \tilde{G}_n \to \tilde{G}'$ is the restriction of $\tilde{\epsilon}_n$ to $\tilde{G}_n$. Then (iii) follows immediately from the definition of f_n.

Now it suffices to construct ϵ_{i-1} satisfying (i) - (iii), assuming that there is an embedding ϵ_i satisfying (i) - (iii). (For the following argument, see Figure 5.) Let $e_1 = [u, v_1]$, $e_2 = [u, v_2]$ denote the oriented edges of G_{i-1} and $e = [u, v]$ denote the oriented edge of G_i, such that μ_i folds e_1, e_2 into e. Let P and Q be the components of $H_i \setminus \Delta_i$ that are dual to u and v, respectively. Let $c_1, c_2, \dots, c_p$ be the edges of G_{i-1}, other than e_1, that end at v_1. Let $d_1, d_2, \dots, d_q$ be the edges of G_{i-1}, other than e_2, that end at v_2. Let $D_{c_1}, D_{c_2}, \dots, D_{c_p}$, $D_{d_1}, D_{d_2}, \dots, D_{d_q}, D_e$ denote the disks of Δ_i that are dual to $c_1, c_2, \dots, c_p$, $d_1, d_2, \dots, d_q, e$, respectively. Then Q is bounded by these disks D_{c_1}, D_{c_2}, $\dots, D_{c_p}, D_{d_1}, D_{d_2}, \dots, D_{d_q}, D_e$. Pick two disjoint meridian disks D_1 and D_2 of H_i parallel to D_e such that D_1 and D_2 are contained in P and such that D_1 and D_e bound a solid cylinder in H_i containing D_2.

Case 1. Suppose that u and v are different vertices of G_i. Then u, v_1, v_2 are different vertices of G_{i-1}, and P, Q are different components of $H_i \setminus \Delta_i$. For each $j \in \{1, 2\}$, let B_j be the union of Q and the solid cylinder in P bounded by D_j and D_e. Then B_j is the 3-disk in H_i bounded by $D_j, D_{c_1}, D_{c_2}, \dots, D_{c_p}$, $D_{d_1}, D_{d_2}, \dots, D_{d_q}$. Choose a (simple) arc α properly embedded in the $(1 + p + q)$-holed sphere

$$\partial B_2 \setminus (D_2 \sqcup D_{c_1} \sqcup D_{c_2} \sqcup \dots \sqcup D_{c_p} \sqcup D_{d_1} \sqcup D_{d_2} \sqcup \dots \sqcup D_{d_q}),$$

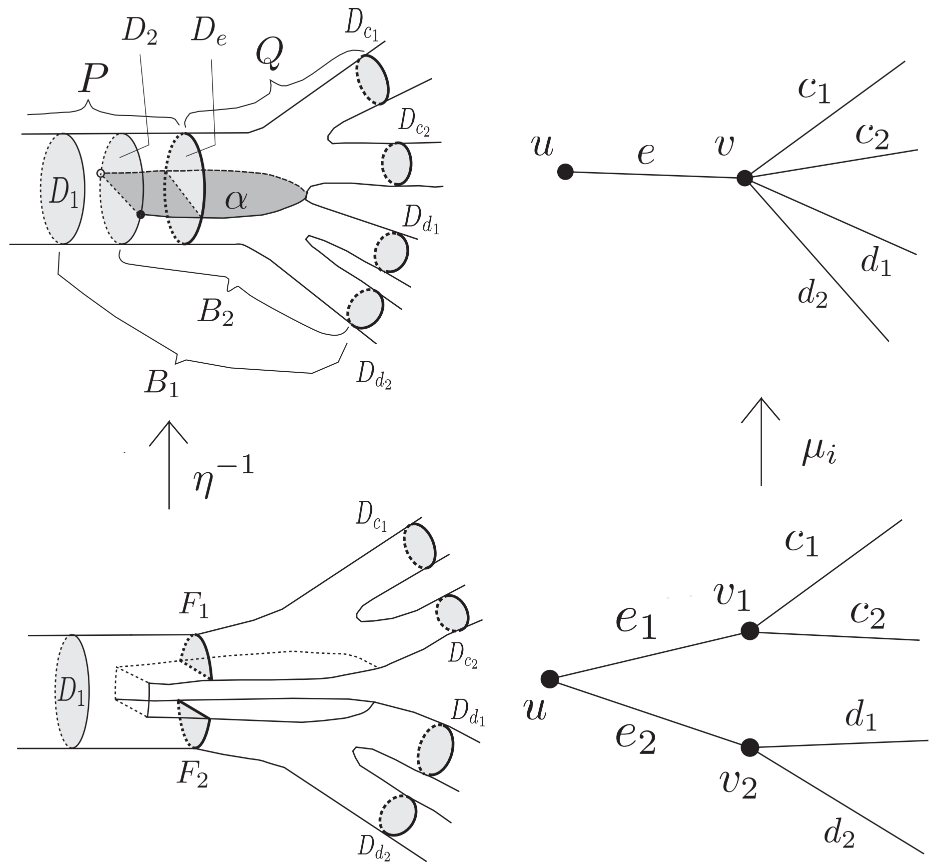

FIGURE 5.

satisfying the following: (I) the ends of α are contained in ∂D_2; (II) α separates $D_{c_1}, D_{c_2}, \ldots, D_{c_p}$ and $D_{d_1}, D_{d_2}, \ldots, D_{d_q}$ in $\partial B_2 \setminus D_2$, and (III) α transversally intersects ∂D_e in exactly two points. Pick an arc β properly embedded in D_2 connecting the end points of α. Then $\alpha \cup \beta$ is a loop on $\partial B_2 \, (\cong \mathbb{S}^2)$. Let E be a 2-disk bounded by $\alpha \cup \beta$ and embedded properly in B_2. In addition, we can assume that E transversally intersects D_e in a single arc. We compress H_i along E as follows (Figure 5): Choose a small regular neighborhood N of E in B_2, so that N splits D_e into two disjoint 2-disks, denoted by F_1 and F_2. Then N is a 3-disk such that $\partial N \cap \partial H_i$ is a 2-disk contained in ∂B_1. Therefore, there is an isotopy η from H_i to $H_i \setminus N$ supported on B_1.

Then $Q \setminus N$ consists of two components. We can assume that a component of $Q \setminus N$ is bounded by $F_1, D_{c_1}, D_{c_2}, \ldots, D_{c_p}$ and the other by $F_2, D_{d_1}, D_{d_2}, \ldots, D_{d_q}$ (if necessarily, by interchanging the symbols F_1 and F_2). Then $H_i \setminus N$ is a genus-g handlebody, and $(\Delta_i \setminus D_e) \cup F_1 \cup F_2 =$

$\Delta_i \cap (H_i \setminus N)$ is a union of meridian disks in the handlebody $H_i \setminus N$. Thus we can see that $(H_i \setminus N, \Delta_i \cap (H_i \setminus N))$ is isomorphic to (H_{i-1}, Δ_{i-1}). Moreover, the isotopy η^{-1} induces the folding map μ_i via duality: v_1^* is the component of $H_{i-1} \setminus \Delta_{i-1}$ bounded by $F_1, D_{c_1}, D_{c_2}, \dots, D_{c_p}$, and v_2^* is the component of $H_{i-1} \setminus \Delta_{i-1}$ bounded by $F_2, D_{d_1}, D_{d_2}, \dots, D_{d_q}$; the isotopy η^{-1} combines F_1 and F_2 into D_e, and accordingly μ_i folds the edges $F_1^* = e_1$ and $F_2^* = e_2$ into $D_e^* = e$; the isotopy η^{-1} preserves all the disks of Δ_{i-1} except F_1 and F_2, and accordingly μ_i preserves all edges of G_{i-1} except e_1 and e_2.

Define $\epsilon_{i-1} \colon (H_{i-1}, \Delta_{i-1}) \to (H', \Delta')$ to be the restriction of $\epsilon_i \colon (H_i, \Delta_i) \to (H', \Delta')$ to $(H_i \setminus N, \Delta_i \cap (H_i \setminus N)) \cong (H_{i-1}, \Delta_{i-1})$. We shall show that ϵ_{i-1} satisfies (i), (ii) and (iii). For every $j = 2, 3$, each j-cell of (H_{i-1}, D_{i-1}) is contained in a j-cell of (H_i, Δ_i). Therefore, since ϵ_i satisfies (i), ϵ_{i-1} also satisfies (i). Via η, ϵ_{i-1} is isotopic to ϵ_i (disregarding the cellular structures). Since ϵ_i satisfies (ii), ϵ_{i-1} also satisfies (ii).

Last, we shall show (iii). Since the isotopy η is supported on the 3-disk B_1 embedded in H_i, it lifts to a (Γ-invariant) isotopy $\tilde{\eta}$ from $\tilde{H}_i$ to $\tilde{H}_{i-1}$ supported on the total lift $\tilde{B}_1$ of B_1 to $\tilde{H}_i$. Since each component R of $\tilde{B}_1$ is homeomorphic to B_1, we can canonically identify $\tilde{\eta}|_R$ with $\eta|_{B_1}$. For each loop ℓ of $\tilde{L}_{i-1}$, let D_ℓ denote the disk of $\tilde{\Delta}_{i-1}$ bounded by ℓ. Let $m = h_i(\ell)$, which is a loop of $\tilde{L}_i$, and let D_m be the disk of $\tilde{\Delta}_i$ bounded by m. Since $f_{i-1} = f_i \circ h_i$, we have $f_{i-1}(\ell) = f_i(h_i(\ell)) = f_i(m)$.

First, suppose that ℓ does *not* bound a lift of F_1 or F_2 to $\tilde{H}_{i-1}$. Since η is the identity on $\Delta_i \setminus D_\ell$, the isotopy $\tilde{\eta}^{-1}$ is the identity on D_m. Therefore $D_\ell = D_m$ via the inclusion $(\tilde{H}_{i-1}, \tilde{\Delta}_{i-1}) \subset (\tilde{H}_i, \tilde{\Delta}_i)$. Then we have $\tilde{\epsilon}_{i-1}(D_\ell) = \tilde{\epsilon}_i(D_m)$. Since $\tilde{\epsilon}_i$ satisfies (iii), we have $\tilde{\epsilon}_i(D_m) \subset Conv(f_i(m)) = Conv(f_{i-1}(\ell))$. Therefore $\tilde{\epsilon}_{i-1}(\ell) \subset Conv(f_{i-1}(\ell))$.

Next, suppose that ℓ bounds a lift of F_1 or F_2. Then, accordingly, D_ℓ is a lift of F_1 or F_2 to $\tilde{H}_{i-1}$. Therefore D_m is a lift of D_e to $\tilde{H}_i$, and it is contained in a component R of $\tilde{B}_1$ via the inclusion $(\tilde{H}_{i-1}, \tilde{\Delta}_{i-1}) \subset (\tilde{H}_i, \tilde{\Delta}_i)$. Since $\tilde{\eta}|_R = \eta|_{B_1}$, via the same inclusion, we have $D_\ell \subset D_m$ and thus $\tilde{\epsilon}_{i-1}(D_\ell) \subset \tilde{\epsilon}_i(D_m)$. As in the first case, we have $\tilde{\epsilon}_i(D_m) \subset Conv(f_i(m)) = Conv(f_{i-1}(\ell))$. Thus $\tilde{\epsilon}_{i-1}(\ell) \subset Conv(f_{i-1}(\ell))$, and therefore ϵ_{i-1} satisfies (iii).

Case 2. (For the following discussion, see Figure 6.) Suppose that u and v are the same vertices of G_n. Then, without loss of generality, we can assume that $u = v_1$ and $u \neq v_2$ (if $u = v_1 = v_2$, there is a contradiction to the definition of a folding map). Then $P = Q$. In addition, we can assume that $e = c_1$. Then $D_e = D_{c_1}$.

Let B_1 be the region in H_i bounded by $D_1, D_{c_2}, \dots, D_{c_p}, D_{d_1}, D_{d_2}, \dots, D_{d_q}$. Let B_2 be the region in H_i bounded by $D_1, D_2, D_{c_2}, \dots, D_{c_p}$,

$D_{d_1}, D_{d_2}, \ldots, D_{d_q}$. Let α be an arc properly embedded in the $(1+p+q)$-holed sphere

$$\partial B_2 \setminus (D_1 \sqcup D_2 \sqcup D_{c_2} \sqcup \ldots \sqcup D_{c_p} \sqcup D_{d_1} \sqcup D_{d_2} \sqcup \ldots \sqcup D_{d_q}),$$

satisfying the following: (I) the end points of α are contained in ∂D_2; (II) α separates $D_1, D_{c_2}, D_{c_3}, \ldots, D_{c_p}$ and $D_{d_1}, D_{d_2}, \ldots, D_{d_q}$ on $\partial B_2 \setminus D_2$, and (III) α transversally intersects $\partial D_e = \partial D_{c_1}$ in exactly two points. The rest of the proof is similar to that of *Case 1*. $\square$

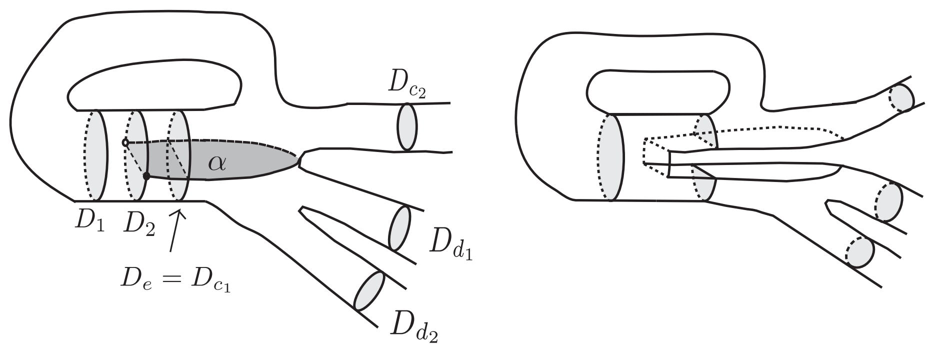

FIGURE 6.

6. DECOMPOSITION OF A SCHOTTKY STRUCTURE INTO GOOD HOLED SPHERES

Recall that we started with a projective surface (S, C) with fuchsian Schottky holonomy ρ. Then, we obtained a multiloop L on S (see §5), and regarded S as the boundary of the handlebody H so that each loop of L bounds a meridian disk in H (see §5.1). Let $\epsilon\colon (H, \Delta) \to (H', \Delta')$ be the embedding obtained by Proposition 5.1. By Proposition 5.1 (ii), there is a homeomorphism $\eta\colon S \times [0,1] \to H' \setminus int(Im(\epsilon))$. Setting $\eta_t(s) = \eta(s,t)$, we can assume that η_0 is a homeomorphism from S to $\epsilon(\partial H) = \epsilon(S)$, that η_1 is a homeomorphism from S to $\partial H' = S'$, and that $\eta_0 = \epsilon|_S$ (via the natural identification of $S \times \{0\}$ with S). Let $M' = \eta_1(L)$, which is a multiloop on S'.

We shall check that M' satisfies Assumptions (I), (II) in Proposition 4.5. First, recall that L satisfies Conclusions (i), (ii), (iii) in Proposition 4.5 (by applying it to $N = L$). Since each component of $S \setminus L$ is a holed sphere and η_1 is a homeomorphism, each component of $S' \setminus M'$ is therefore a holed sphere as well ((II)). For each loop m' of M', let $\ell = \eta_1^{-1}(m')$. Then ℓ is a loop of L and $\eta(\ell \times [0,1])$ is an annulus

properly embedded in $H' \setminus int(Im(\epsilon))$ bounded by $\eta_1(\ell) = m'$ and $\eta_0(\ell) = \epsilon(\ell)$. Since ℓ bounds a meridian disk in H, $\epsilon(\ell)$ also bounds a meridian disk in $Im(\epsilon)$. Thus the union of this meridian disk in $Im(\epsilon)$ and the annulus $\eta(\ell \times [0,1])$ is a meridian disk in H' bounded by m' ((II)).

Let M be the pullback of M' via f, which is a multiloop on S (see §4.1). In particular, Proposition 4.5 (iii) asserts that M decomposes (S, C) into almost good holed spheres. In this section, we prove the following theorem stating that M decomposes (S, C) even into good holed spheres: Let $\tilde{M}$ and $\tilde{M}'$ denote the total lifts of M and M' to $\tilde{S}$ and Ω, respectively.

Theorem 6.1. *If P is a component of $S \setminus M$, then $C|P$, the restriction of C to P, is a good holed sphere fully supported on a component of $\Omega \setminus \tilde{M}$. Moreover, there exists a ρ-equivariant homeomorphism $\zeta : \tilde{S} \to \Omega$ such that, if $\tilde{P}$ is a component of $\tilde{S} \setminus \tilde{M}$, then*
(i) each boundary component ℓ of $\tilde{P}$ covers $\zeta(\ell)$ via f; therefore
(ii) $\tilde{C}|\tilde{P}$ is a good holed sphere fully supported on $\zeta(\tilde{P})$, where $\tilde{C}$ is the projective structure on $\tilde{S}$ obtained by lifting C.

Theorem 6.1 follows from:

Proposition 6.2. *If μ' is a loop of $\tilde{M}'$, then $\lfloor f^{-1}(\mu') \rfloor$ is a single loop on $\tilde{S}$.*

Proof of Theorem 6.1 (i), with Proposition 6.2 assumed. By Proposition 6.2, there is a one-to-one correspondence between the loops of $\tilde{M}$ and the loops of $\tilde{M}'$ via f. Therefore, we can choose a ρ-equivariant homeomorphism $\zeta \colon \tilde{M} \to \tilde{M}'$ such that $f(m) = \zeta(m)$ for each loop m of $\tilde{M}$; indeed, we can first define ζ on some loops of $\tilde{M}$ whose union is a fundamental domain for the Γ-action on $\tilde{M}$ and then extend η ρ-equivariantly to a homeomorphism from $\tilde{M}$ onto $\tilde{M}'$. If P is a component of $\tilde{S} \setminus \tilde{M}$, then $\tilde{C}|P$ is an almost good holed sphere whose support is a unique component R of $\Omega \setminus \tilde{M}'$. By Corollary 4.6, if two loops a, b of $\tilde{M}$ are boundary components of a single component of $\tilde{S} \setminus \tilde{M}$, then $f(a), f(b)$ are also boundary components of a single component of $\Omega \setminus \tilde{M}'$. By Proposition 6.2 and the equivariancy of η, we see that $\zeta|_{\partial P}$ must be a homeomorphism onto ∂R. Therefore $\tilde{C}|P$ is a good holed sphere fully supported on R, and P is homeomorphic to R. Since $\zeta : \tilde{M} \to \tilde{M}'$ is a homeomorphism, the components of $\tilde{S} \setminus \tilde{M}$ bijectively correspond to the components of $\Omega \setminus \tilde{M}'$ (as supports). Therefore we can extend ζ to a ρ-equivariant homeomorphism from $\tilde{S}$ to Ω. $\square$

6.1. **An outline of the proof of Proposition 6.2.** Proposition 6.2 is the main proposition of this paper, and we here outline its (lengthy) proof. Let λ be the loop of $\tilde{M}$ with $\tilde{\eta}_1(\lambda) = \mu'$, and let λ' be the loop of $\tilde{M}'$ with $f(\lambda) = \lambda'$.

Step 1. The proof will be reduce to a statement similar to the proposition, but, regarding to a certain good holed sphere related to $\tilde{C}$, as follows: We first show that $\lfloor f^{-1}(\mu') \rfloor$ is a multiloop contained in a compact subsurface F of $\tilde{S}$, such that $F \supset \lambda$ and $\tilde{C}|F$ is an almost good holed sphere (Proposition 6.4 and Corollary 6.5). We next extend $\tilde{C}|F$ to a good structure $C_{\check{F}} = (f_{\check{F}}, \rho_{id})$ on a punctured sphere $\check{F}$ so that $\lfloor f^{-1}(\mu') \rfloor = \lfloor f_{\check{F}}^{-1}(\mu') \rfloor \subset F$. Thus it suffices to show that $\lfloor f_{\check{F}}^{-1}(\mu') \rfloor$ is a single loop (Proposition 6.8).

Step 2. Recall that Proposition 5.1 yields the embedding $\tilde{\epsilon}\colon \tilde{H} \to \overline{\mathbb{H}^3}$ that corresponds to $f\colon \tilde{S} \to \hat{\mathbb{C}}$. We construct an analogous embedding that corresponds to $f_{\check{F}}\colon \check{F} \to \hat{\mathbb{C}}$, as follows. Let $P_{\check{F}}$ denote the punctures of $\check{F}$. Let $\hat{F}$ be the 2-sphere $\check{F} \cup P_{\check{F}}$, and let $H_{\hat{F}}$ be the 3-disk with $\partial H_{\hat{F}} = \hat{F}$. We construct an embedding $\epsilon_{\hat{F}}\colon H_{\hat{F}} \to \overline{\mathbb{H}^3}$ satisfying the following conditions:
(i) $\epsilon_{\hat{F}}(p) = f_{\check{F}}(p) \in \hat{\mathbb{C}}$ for each $p \in P_{\check{F}}$, and $\epsilon_{\hat{F}}(H_{\hat{F}} \setminus P_{\check{F}}) \subset \mathbb{H}^3$,
(ii) $\overline{\mathbb{H}^3} \setminus Im(\epsilon_{\hat{F}})$ is homeomorphic to $\check{F} \times [0,1]$, and
(iii) $Im(\epsilon_{\hat{F}}) \cap Conv(\lambda')$ is a union of disjoint 2-disks, one of which is bounded by $\epsilon_{\check{F}}(\lambda)$, and $\lfloor f_{\check{F}}^{-1}(\lambda') \rfloor = \partial \epsilon_{\hat{F}}^{-1}(Conv(\lambda'))$ (see Lemma 6.9).

Step 3. We construct a homeomorphism $\phi\colon \overline{\mathbb{H}^3} \to \overline{\mathbb{H}^3}$, by extending a certain isotopy $Im(\epsilon_{\hat{F}})$ in $\mathbb{H}^3$, such that $\phi(Im(\epsilon)_{\check{F}})) \cap Conv(\lambda')$ is a single disk bounded by $\phi(\epsilon_{\check{F}}(\lambda))$ ($= \epsilon_\phi(\lambda)$; see Figure 9). From the correspondence between $f_{\check{F}}$ and $\epsilon_{\hat{F}}$, it follows that $\lfloor (\phi \circ f_{\check{F}})^{-1}(\lambda') \rfloor = \lfloor f_{\check{F}}^{-1}(\phi^{-1}(\lambda')) \rfloor$ is, accordingly, a single loop on $\check{F}$ and that $\phi^{-1}(\lambda')$ is isotopic to μ' in the punctured sphere $\hat{\mathbb{C}} \setminus \epsilon_{\hat{F}}(P_{\check{F}})$. Then $\lfloor f^{-1}(\mu') \rfloor$ is a single loop isotopic to $\lfloor (\phi \circ f_{\check{F}})^{-1}(\lambda') \rfloor$.

6.2. **The proof of Proposition 6.2.** *Step 1.* Recall also that Ω_0 is the compact fundamental domain for the Γ-action on Ω bounded by $2g$ round loops of $\tilde{L}'$. Accordingly $Conv(\Omega_0)$ is a fundamental domain for the Γ-action on $\mathbb{H}^3 \cup \Omega$. Then $Conv(\Omega_0)$ is a compact subset of $\mathbb{H}^3 \cup \Omega$ bounded by $2g$ copies of $\overline{\mathbb{H}^2}$ that are disks of $\tilde{\Delta}'$. Recall that $\tilde{\epsilon}$ is the lift of ϵ to a $\tilde{\rho}$-equivariant embedding of $\tilde{H}$ into $\mathbb{H}^3 \cup \Omega$. We let $\tilde{\eta} : \tilde{S} \times [0,1] \to (\mathbb{H}^3 \cup \Omega) \setminus int(Im(\tilde{\epsilon}))$ denote the $\tilde{\rho}$-equivariant homeomorphism obtained by lifting η. Set $\tilde{\eta}_t(s) = \tilde{\eta}(s,t)$.

Let μ' be a loop of $\tilde{M}'$ and λ be its corresponding loop of $\tilde{L}$, i.e. $\tilde{\eta}_1(\lambda) = \mu'$. Recall that $f(\lambda)$ is a loop of $\tilde{L}'$ such that $Conv(f(\lambda)) \supset$

$\tilde{\epsilon}(\lambda)$ (Proposition 5.1 (iii)). Then $\tilde{\eta}(\lambda \times [0,1])$ is a compact annulus that is embedded properly in $(\mathbb{H}^3 \cup \Omega) \setminus int(Im(\tilde{\epsilon})) \cong \tilde{S} \times [0,1]$ and bounded by $\tilde{\eta}(\lambda \times \{0\}) = \tilde{\epsilon}(\lambda)$ and $\tilde{\eta}(\lambda \times \{1\}) = \mu'$. Let

$$F_0' = \bigcup \{\, \gamma\Omega_0 \,|\, \gamma \in \Gamma,\, Conv(\gamma\Omega_0) \cap \tilde{\eta}(\lambda \times [0,1]) \neq \emptyset \,\}.$$

Lemma 6.3. *F_0' is a compact connected subsurface of Ω bounded by finitely many loops of $\tilde{L}'$, and the interior of F_0' contains μ' and $f(\lambda)$.*

Proof. For each $\gamma \in \Gamma$, $Conv(\gamma\Omega_0)$ is the closure of a component of $(\mathbb{H}^3 \cup \Omega) \setminus \tilde{\Delta}'$, which is a compact region in $\mathbb{H}^3 \cup \Omega$ bounded by $2g$ disks of $\tilde{\Delta}'$. Since $\tilde{\eta}(\lambda, [0,1])$ is compact, $\tilde{\eta}(\lambda \times [0,1])$ intersects $Conv(\gamma\Omega_0)$ for only finitely many $\gamma \in \Gamma$. Therefore, letting

$$E = \cup\{\, Conv(\gamma\Omega_0) \,|\, \gamma \in \Gamma,\, Conv(\gamma\Omega_0) \cap \tilde{\eta}(\lambda, [0,1]) \neq \emptyset \,\},$$

E is compact. Besides, since $\tilde{\eta}(\lambda \times [0,1])$ is connected, E is also connected. Thus E is a connected compact convex subset of $\mathbb{H}^3 \cup \Omega$ bounded by (finitely many) disks of $\tilde{\Delta}'$. Since $Conv(F_0') = E$, F_0' is a connected compact subsurface of Ω bounded by finitely many loops of $\tilde{L}'$. Since $\mu' = \tilde{\eta}(\lambda, \{1\}) \subset \tilde{\eta}(\lambda, [0,1])$, the interior of F_0' contains μ' by the definition of F_0'. Since $\tilde{\eta}(\lambda, \{0\}) = \epsilon(\lambda) \subset Conv(f(\lambda))$, F_0' contains both components of $\Omega \setminus \tilde{L}'$ that have $f(\lambda)$ as a boundary component. Therefore the interior of F_0' also contains $f(\lambda)$. □

Proposition 6.4. *There exist a compact connected subsurface F of $\tilde{S}$ bounded by finitely many loops of $\tilde{L}$ and a compact connected subsurface F' of Ω bounded by finitely many loops of $\tilde{L}'$, which satisfy the following properties:*
(i) F' contains F_0',
(ii) $\tilde{C}|F$ is an almost good holed sphere supported on F', and
(iii) F contains $\tilde{\epsilon}^{-1}(Conv(F_0')) \cap \tilde{S}$.

Proof. For a component R of $\tilde{S} \setminus \tilde{L}$, we have either $R \subset F_0'$ or $R \subset \tilde{S} \setminus F_0'$. By Corollary 5.2, if $R \subset F_0'$, then $\tilde{\epsilon}(R) \subset Conv(F_0')$ and, if $R \subset \tilde{S} \setminus F_0'$, then $\tilde{\epsilon}(R) \cap Conv(F_0') = \emptyset$. Therefore we have

$$\tilde{\epsilon}^{-1}(Conv(F_0')) \cap \tilde{S} = \bigcup \{cl(R) \mid R \subset F_0',\ R \textit{ is a component of } \tilde{S} \setminus \tilde{L}\}.$$

Since f is $\tilde{\rho}$-equivariant, for each $\gamma \in \Gamma$, there is at least one but at most finitely many components of $\tilde{S} \setminus \tilde{L}$ supported on $\gamma\Omega_0$. Therefore $\tilde{\epsilon}^{-1}(Conv(F_0')) \cap \tilde{S}$ is a compact subsurface of $\tilde{S}$ bounded by finitely many loops of $\tilde{L}$. Note that $\tilde{\epsilon}^{-1}(Conv(F_0')) \cap \tilde{S}$ is *not* necessarily connected. Thus we can choose a compact connected subsurface F_0 of $\tilde{S}$ bounded by finitely many loops of $\tilde{L}$ such that $F_0 \supset \tilde{\epsilon}^{-1}(Conv(F_0')) \cap \tilde{S}$.

Each component Q of $F_0 \setminus \tilde{L}$ is a component of $\tilde{S} \setminus \tilde{L}$, and $\tilde{C}|Q$ is supported on a unique component of $\Omega \setminus \tilde{L}'$. Let

$$F' = \bigcup cl(\, Supp(\, \tilde{C}|Q)\,),$$

where Q varies over all components of $F_0 \setminus \tilde{L}$. By the definitions of F_0 and F', F' contains F'_0 ((i)). Since F_0 is a compact connected subsurface of $\tilde{S}$ bounded by finitely many loops of $\tilde{L}$, using Corollary 4.6, we can see that F' is a compact connected subsurface of Ω bounded by finitely many loops of $\tilde{L}'$. In particular, F' is a holed sphere in Ω. By a similar argument, $\tilde{\epsilon}^{-1}(Conv(F')) \cap \tilde{S}$ is the union of finitely many components R of $\tilde{S} \setminus \tilde{L}$ such that $Supp(\tilde{C}|R) \subset F'$ and the loops of $\tilde{L}$ bounding all such R. Then $\tilde{\epsilon}^{-1}(Conv(F')) \cap \tilde{S}$ is a compact subsurface of $\tilde{S}$ bounded by finitely many loops of $\tilde{L}$, but again it is *not* necessarily connected. By the definition of F', $\tilde{\epsilon}^{-1}(Conv(F')) \cap \tilde{S}$ contains a component that contains F_0. Let F denote this component. Since F_0 contains $\tilde{\epsilon}^{-1}(Conv(F'_0)) \cap \tilde{S}$, F also contains $\tilde{\epsilon}^{-1}(Conv(F'_0)) \cap \tilde{S}$ ((iii)). Since F is a compact, connected and planar subsurface of $\tilde{S}$ bounded by finitely many loops of $\tilde{L}$, F is a sphere with at least two holes. By Proposition 5.1 (iii) and Corollary 4.6, ∂F covers $\partial F'$ via f and, since F is *not* a 2-disk, $f(\partial F)$ must be a union of at least two components of $\partial F'$. Therefore $\tilde{C}|F$ is an almost good holed sphere supported on F' ((ii)). $\square$

Corollary 6.5. *F, as in Proposition 6.4, contains $\lfloor f^{-1}(\mu') \rfloor$.*

Proof. Take an arbitrary component of $\tilde{S} \setminus \tilde{L}$ that is disjoint from F. Then, let R denote the closure of this component. It suffices to show that $R \cap \lfloor f^{-1}(\mu') \rfloor = \emptyset$. By Proposition6.5 (iii), for each component Q of $R \setminus \tilde{L}$, $\tilde{\epsilon}(Q) \cap Conv(F'_0) = \emptyset$. Then, by Corollary 5.2, $Supp_f(Q) \cap int(F'_0) = \emptyset$. Then $Supp_f(Q)$ is disjoint from $int(F'_0)$, and hence $Supp_f(R)$ is also disjoint from $int(F'_0)$. Then, by Lemma 6.3, μ' and $Supp_f(R)$ are disjoint. Hence, by Lemma 4.3, $R \cap \lfloor f^{-1}(\mu') \rfloor = \emptyset$. $\square$

Step2. Set $\tilde{C}|F = (f_F, \rho_{id})$. In order to prove Proposition 6.2, by Corollary 6.5, it suffices to show that $\lfloor f_F^{-1}(\mu') \rfloor = \lfloor f^{-1}(\mu') \rfloor \cap F$ is a single loop on F. The boundary components of F bound some disks of $\tilde{\Delta}$ in $\tilde{H}$. The union of such disks bound a 3-disk, H_F, in $\tilde{H}$, so that $H_F \cap \tilde{S} = F$. Let $\epsilon_F : H_F \to \mathbb{H}^3$ denote the restriction of $\tilde{\epsilon}$: $\tilde{H} \to \mathbb{H}^3$ to H_F. Accordingly, restricting the homeomorphism $\tilde{\eta}$: $\tilde{S} \times [0,1] \to \overline{\mathbb{H}^3} \setminus int(Im(\tilde{\epsilon}))$ to $F \times [0,1]$, we obtain a homeomorphism $\eta_F : F \times [0,1] \to \tilde{\eta}(F \times [0,1])$. Let $\check{F}$ denote the punctured sphere obtained by

attaching a once-punctured disk along each boundary component of F. Let $p_1, p_2, \ldots, p_n$ denote the punctures of $\check{F}$, where n is the number of the boundary components of F. Then $\check{F} \cup p_1 \cup p_2 \ldots \cup p_n =: \hat{F}$ is a 2-sphere, and we regard $p_1, p_2, \ldots, p_n$ as (distinct) marked points on $\hat{F}$. Let $H_{\hat{F}}$ be a closed 3-disk and regard $\partial H_{\hat{F}}$ as $\hat{F}$. Then $\partial F\,(\subset \hat{F})$ bounds disjoint 2-disks properly embedded in $H_{\hat{F}}$. Then the union of these disjoint disks bounds a 3-disk in $H_{\hat{F}}$ that can be naturally identified with H_F.

Next we extend $\epsilon_F \colon H_F \to \overline{\mathbb{H}^3}$ to an embedding $\epsilon_{\hat{F}} \colon H_{\hat{F}} \to \overline{\mathbb{H}^3}$ so that $\overline{\mathbb{H}^3} \setminus (\,int(Im(\epsilon_{\hat{F}})) \cup \{\epsilon_{\hat{F}}(p_1), \epsilon_{\hat{F}}(p_2), \ldots, \epsilon_{\hat{F}}(p_n)\}\,)$ is homeomorphic to $\check{F} \times [0,1]$ and, with respect to this identification, we have $\epsilon_{\hat{F}}(\check{F}) \cong \check{F} \times \{0\}$ and $\hat{\mathbb{C}} \setminus \{\epsilon_{\hat{F}}(p_1), \epsilon_{\hat{F}}(p_2), \ldots, \epsilon_{\hat{F}}(p_n)\} \cong \check{F} \times \{1\}$. Each boundary component ℓ of F bounds a disk D_ℓ of $\tilde{\Delta}$ in $\tilde{H}$, which can be identified with one of the disks bounding H_F in $H_{\hat{F}}$. Let F_ℓ be the (unbounded) component of $\tilde{S} \setminus F$ bounded by ℓ. Then $F_\ell \setminus \tilde{L}$ is a union of infinitely many components of $\tilde{S} \setminus \tilde{L}$. Recalling the inclusion $H_{\hat{F}} \supset H_F$, let H_ℓ be the component of $H_{\hat{F}} \setminus H_F$ bounded by D_ℓ. Then H_ℓ is (topologically) a 3-disk whose boundary sphere is the union of D_ℓ and the component of $\hat{F} \setminus F$ bounded by ℓ. The component of $\hat{F} \setminus F$ is a 2-disk whose interior contains a single marked point $p_\ell \in \{p_1, p_2, \ldots, p_n\}$ corresponding to ℓ.

In $\overline{\mathbb{H}^3}$, $Conv(f(\ell)) \cong \overline{\mathbb{H}^2}$ is a boundary component of $Conv(F')$. Then, let X_ℓ be the component of $\overline{\mathbb{H}^3} \setminus Conv(F')$ bounded by $Conv(f(\ell))$. Let Y_ℓ be the component of $\hat{\mathbb{C}} \setminus F'$ bounded by $f(\ell)$, so that $X_\ell \cap \hat{\mathbb{C}} = Y_\ell$.

Lemma 6.6. *There is a sequence $(R_i)_{i=1}^{\infty}$ of distinct connected components of $F_\ell \setminus \tilde{L}$ such that*
(i) ℓ is a boundary component of R_1,
(ii) R_i and R_{i+1} are adjacent subsurfaces of F_l for all $i = 1, 2, 3, \ldots$,
(iii) $\tilde{\epsilon}(R_i) \subset X_\ell$ for all $i = 1, 2, 3, \ldots$, and
(iv) $(\tilde{\epsilon}(R_i))_{i=1}^{\infty}$ limits to a limit point of Γ (contained in Y_ℓ).

Proof. Let R_1 be the component of $F_\ell \setminus \tilde{L}$ bounded by $\ell =: \ell_0$ ((i)). Then $\tilde{C}|R_i$ is a good holed sphere supported on a unique component of $\Omega \setminus \tilde{L}'$. Let Ω_1 be this component. By Corollary 4.6, Ω_1 and F' are adjacent subsurfaces of Ω, sharing $f(\ell)$ as a boundary component. Thus $\Omega_1 \subset Y_\ell$, and by Proposition 5.1, $\tilde{\epsilon}(R_1) \subset Conv(\Omega_1) \subset X_\ell$.

We shall inductively define R_i for $i \geq 2$. Assume that we have picked components $R_1, R_2, \ldots, R_i$ satisfying (ii) with R_1 defined above. Then, let Ω_i be the component of $\Omega \setminus \tilde{L}'$ on which $\tilde{C}|R_i$ is supported. Let $\ell_{i-1}(\subset \tilde{L})$ denote the common boundary component of R_{i-1} and R_i. By the definition of an almost good holed sphere, $f(\partial R_i)$ is a union

of at least 2 boundary components of Ω_i. Therefore we can pick a boundary component ℓ_i of R_i such that $f(\ell_i)$ and $f(\ell_{i-1})$ are different boundary components of Ω_i. Let R_{i+1} be the component of $F_\ell \setminus \tilde{L}$ adjacent to R_i, sharing ℓ_i as a boundary component.

Next we shall show

Claim 6.7. *For each $k \geq 1$, $\Omega_1, \Omega_2, \ldots, \Omega_k$ are distinct components of $Y_\ell \setminus \tilde{L}'$, and $cl(\sqcup_{i=1}^k \Omega_i)$ is a holed sphere in Y_ℓ bounded by finitely many loops of $\tilde{L}'$ such that there is a 2-disk component of $Y_\ell \setminus cl(\sqcup_{i=1}^k \Omega_i)$ bounded by $f(\ell_k)$.*

Proof. (See Figure 7.) Clearly this claim holds for $k = 1$. Assume that this claim holds for some $k \geq 1$. Let B be the 2-disk component of $Y_\ell \setminus cl(\sqcup_{i=1}^k \Omega_i)$ bounded by $f(\ell_k)$. By the construction of $(R_i)_{i=1}^\infty$, Ω_k and Ω_{k+1} are adjacent components, sharing $f(\ell_k)$ as a boundary component. Then Ω_{k+1} is a sphere with at least 2 holes contained in B. Therefore $\Omega_1, \Omega_2, \ldots, \Omega_k, \Omega_{k+1}$ are distinct components of $Y_\ell \setminus \tilde{L}'$, and $cl(\sqcup_{i=1}^{k+1} \Omega_i)$ is again a holed sphere in Y_ℓ bounded by finitely many loops of $\tilde{L}'$. Since $f(\ell_k)$ and $f(\ell_{k+1})$ are different boundary components of Ω_{k+1}, $B \setminus \Omega_{k+1}$ contains a 2-disk component bounded by ℓ_{k+1}. This component is the component of $Y_\ell \setminus cl(\sqcup_{i=1}^{k+1} \Omega_i)$ bounded by $f(\ell_k)$. ☐

By Claim 6.7, $\Omega_i \subset Y_\ell$ for all $i \geq 1$. Therefore $\tilde{\epsilon}(R_i) \subset Conv(\Omega_i) \subset X_\ell$ ((iii)). Claim 6.7 also implies that $(f(\ell_i))_{i=1}^\infty$ is a sequence of *nested* loops of $\tilde{L}' \cap Y_\ell$, i.e. $Y_\ell \setminus cl(\sqcup_{i=1}^\infty f(\ell_i))$ is a union of disjoint cylinders bounded by $f(\ell_{i-1})$ and $f(\ell_i)$, where $i = 1, 2, \ldots$ (see Figure 7.1). Since $\tilde{L}'$ splits Ω into fundamental domains for Γ, $(f(\ell_i))_{i=1}^\infty$ limits to a limit point of Γ contained in Y_ℓ. Since $\tilde{\epsilon}(R_i) \subset Conv(\Omega_i)$ and Ω_i is bounded by $f(\ell_{i-1})$ and $f(\ell_i)$, therefore $(\tilde{\epsilon}(R_i))_{i=1}^\infty$ limits to the same limit point ((iv)). ☐

Let G_ℓ be the closure of the union of R_i, over $i = 1, 2, \ldots$, obtained by Lemma 6.6. Then G_ℓ is an unbounded connected subsurface of F_l $(\subset \tilde{S})$ bounded by infinitely many loops of $\tilde{L}$. Then ∂G_ℓ is a multiloop on $\tilde{S}$ bounding disks of $\tilde{\Delta}$ in $\tilde{H}$. Let $H(G_\ell)$ be the closed subset of $\tilde{H}$ bounded by these disks of $\tilde{\Delta}$, so that $G_\ell = H(G_\ell) \cap \tilde{S}$. Then $H(G_\ell)$ is homeomorphic to $\mathbb{D}^3$ minus a point in $\partial \mathbb{D}^3$ corresponding to the limit point in Lemma 6.6 (iv). Therefore $H(G_\ell)$ can be naturally identified with H_ℓ minus the marked point p_l in $\hat{F} \cap H_\ell$. Note that the domain of $\tilde{\epsilon}$ contains $H(G_\ell)$. In addition, $\tilde{\epsilon}$ continuously extends to the end point of $\tilde{S}$ corresponding to p_ℓ, so that this end point maps to the limit point of Γ for ℓ in Lemma 6.6. Now the embedding $\epsilon_F \colon H_F \to \overline{\mathbb{H}^3}$ has extended to an embedding of $H_F \cup H_\ell$ into $\overline{\mathbb{H}^3}$ via $\tilde{\epsilon}$. By Lemma 6.6

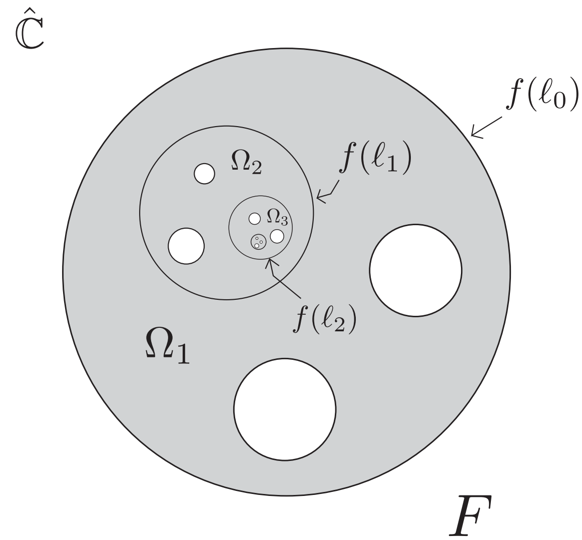

FIGURE 7. The shaded region is $cl(\sqcup_{i=1}^{k}\Omega_i)$.

(iii), for different boundary components ℓ of F, corresponding $H(G_\ell)$ are contained in some different components of $\tilde{H} \setminus H_F$. Therefore, such extension yields an embedding $\epsilon_{\hat{F}}\colon H_{\hat{F}} \to \overline{\mathbb{H}^3}$. Note that $\epsilon_{\hat{F}}$ takes $H_{\hat{F}} \setminus \{p_1, p_2, \dots, p_n\}$ to $\mathbb{H}^3$ and $p_1, p_2, \dots, p_n$ to different points on $\hat{\mathbb{C}}$ (note that $\tilde{\epsilon}$ is ρ-equivariant).

Next we shall show that $\overline{\mathbb{H}^3} \setminus int(Im(\epsilon_{\hat{F}}))$ has a natural product structure. Let $\bar{F} = F \cup (\sqcup G_\ell)$, where the union runs over all boundary components ℓ of F. Then $\bar{F}$ is a connected subsurface of $\tilde{S}$ bounded by infinitely many loops of $\tilde{L}$. By the identification of $H(G_\ell)$ and $H_\ell \setminus p_\ell$, the inclusion $F \subset \check{F}$ extends to the inclusion $\bar{F} \subset \check{F}$. Then $\partial\bar{F}$ bounds infinitely many disks of $\tilde{\Delta}$, and $\check{F} \setminus \bar{F}$ is a union of infinitely many disjoint 2-disks corresponding to these disks of $\tilde{\Delta}$. For a boundary component m of $\bar{F}$, let D_m denote the disk of $\tilde{\Delta}$ bounded by m. (For the following discussion, see Figure 8.) Then $\tilde{\epsilon}(D_m)$ is a 2-disk and $\tilde{\eta}(m \times [0,1])$ is an annulus embedded in $\overline{\mathbb{H}^3}$. Since $\tilde{\epsilon}(D_m)$ and $\tilde{\eta}(m \times [0,1])$ share a boundary component, their union $\tilde{\epsilon}(D_m) \cup \tilde{\eta}(m \times [0,1]) =: E'_m$ is a 2-disk properly embedded in $\overline{\mathbb{H}^3}$. Then we have the multidisk $\sqcup E'_m$, where m runs over all boundary components of $\bar{F}$. We can see that $\sqcup E'_m$ bounds $\tilde{\eta}(\bar{F} \times [0,1]) \cup Im(\epsilon_{\hat{F}})$, which is homeomorphic to a closed 3-disk.

For each boundary component m of $\bar{F}$, let D'_m be the (disk) component of $\hat{\mathbb{C}} \setminus \tilde{\eta}(\bar{F} \times \{1\})$, bounded by $\tilde{\eta}(m \times \{1\})$. Then D'_m and

E'_m share $\tilde{\eta}(m \times \{1\})$ as a boundary component, and $D'_m \cup E'_m$ is a 2-sphere. Let Q'_m be the 3-disk in $\overline{\mathbb{H}^3}$ bounded by $D'_m \cup E'_m$. Then Q'_m is a component of $\overline{\mathbb{H}^3} \setminus (\tilde{\eta}(\bar{F} \times [0,1]) \cup Im(\epsilon_{\hat{F}}))$, bounded by E'_m. Choose a homeomorphism $\eta_m \colon D_m \times [0,1] \to Q'_m$ such that $\eta_m(D_m \times \{0\}) = \tilde{\epsilon}(D_m)$, $\eta_m(D_m \times \{1\}) = D'_m$ and $\eta_m|_{\partial D_m \times [0,1]} = \tilde{\eta}|_{m \times [0,1]}$. Then let $\eta_{\bar{F}} : \bar{F} \times [0,1] \to \overline{\mathbb{H}^3}$ denote the restriction of $\tilde{\eta} : \tilde{S} \times [0,1] \to \overline{\mathbb{H}^3}$ to $\bar{F} \times [0,1]$. Then, by naturally identifying $\partial D_m \times [0,1]$ and $m \times [0,1]$ for all boundary components m of $\bar{F}$, we can extend $\eta_{\bar{F}}$ to a homeomorphism

$$\eta_{\check{F}} \colon \check{F} \times [0,1] \to \overline{\mathbb{H}^3} \setminus [\, int(Im(\epsilon_{\hat{F}})) \cup (\sqcup_{i=1}^{n} \epsilon_{\hat{F}}(p_i)) \,]$$

such that $\eta_{\check{F}}(\check{F} \times \{0\}) = \epsilon_{\hat{F}}(\check{F})$ and $\eta_{\check{F}}(\check{F} \times \{1\}) = \hat{\mathbb{C}} \setminus \sqcup_{i=1}^{n} \epsilon_{\hat{F}}(p_i)$.

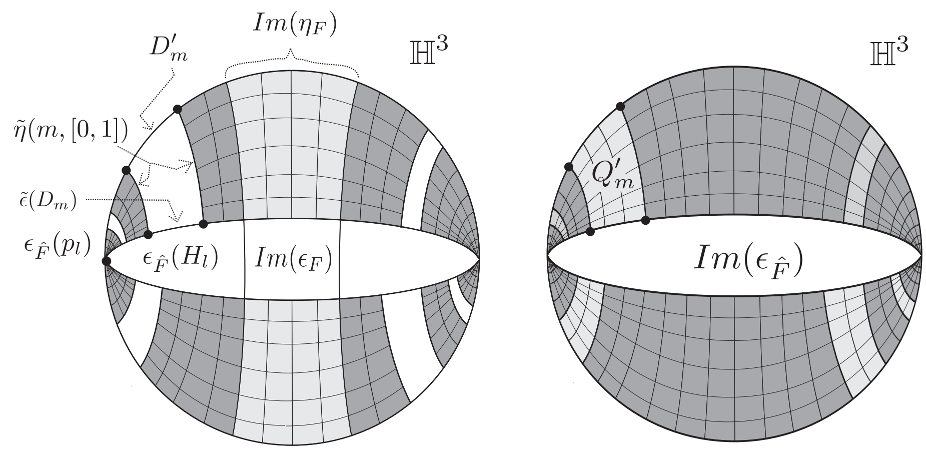

Figure 8. A schematic for the product structure given by $\eta_{\check{F}}$. On the left, the darkest region corresponds to $\tilde{\eta}(G_l \times [0,1])$'s, and, On the right, to $\tilde{\eta}(\bar{F} \times [0,1])$.

Next we shall extend the almost good structure $\tilde{C}|F = (f|_F, \rho_{id})$ on the holed sphere F supported on F' to a good structure on the punctured sphere $\check{F}$. Recall that each boundary component ℓ of F bounds a component of $\hat{F} \setminus F$, which is a 2-disk with the puncture point p_ℓ in its interior. In addition, $\epsilon_{\hat{F}}(p_\ell)$ is contained in the component of $\hat{\mathbb{C}} \setminus F'$ bounded by $f(\ell)$. Recall also that $\epsilon_{\hat{F}}$ takes different punctures of $\check{F}$ to different points on $\hat{\mathbb{C}}$. Therefore, as in §3.4, we can uniquely extend the almost good projective structure $\tilde{C}|F$ on F to a good projective structure $C_{\check{F}} = (f_{\check{F}}, \rho_{id})$ on $\check{F}$ such that $f_{\check{F}}(p_\ell) = \epsilon_{\hat{F}}(p_\ell)$ for each boundary component ℓ of F. Thus $Supp(C_{\check{F}})$ is $\hat{\mathbb{C}} \setminus \sqcup_{i=1}^{n} \epsilon_{\hat{F}}(p_i)$.

Recall that, in order to prove Proposition 6.2, it suffices to show that $\lfloor f_F^{-1}(\mu') \rfloor = \lfloor f^{-1}(\mu') \rfloor \cap F$ is a single loop on F (see the discussion after

Corollary 6.5). If X is a component of $\check{F}\backslash F$, then X covers a component of $\hat{\mathbb{C}} \setminus F'$ minus a point via $f_{\check{F}}$. Since $F' \supset \mu'$, $f_{\check{F}}^{-1}(\mu') \cap X = \emptyset$. Thus the proof of Proposition 6.2 is reduced to:

Proposition 6.8. $\lfloor f_{\check{F}}^{-1}(\mu') \rfloor$ *is a single loop on* $\check{F}$.

Recall that λ and μ' are the loops on $\hat{F}$ and $\hat{\mathbb{C}}$, respectively, such that $\mu' = \tilde{\eta}(\lambda \times \{1\}) = \eta_{\check{F}}(\lambda \times \{1\})$. Let $\lambda' = f(\lambda)$. Then λ' is the loop of $\tilde{L}$ satisfying $\epsilon_{\hat{F}}(\lambda) \subset Conv(\lambda') \cong \overline{\mathbb{H}^2}$. Let $D'_{\lambda'} = Conv(\lambda')$.

Lemma 6.9. $\epsilon_{\hat{F}}^{-1}(D'_{\lambda'})$ *is a multidisk properly embedded in* $H_{\hat{F}}$ *bounded by the multiloop* $\lfloor f_{\check{F}}^{-1}(\lambda') \rfloor$.

Proof. Recall that H_F and all H_ℓ have disjoint interiors, where ℓ are all boundary components of F, and that $H_{\hat{F}}$ is the union of H_F and all H_ℓ. By Lemma 6.3, λ' is contained in $int(F'_0) \subset int(F')$. Then $D'_{\lambda'}$ is contained in $int(Conv(F'))$. Thus, since X_ℓ is a component of $\overline{\mathbb{H}^3} \setminus Conv(F')$ and we have $\epsilon_{\hat{F}}(H_\ell) \subset X_\ell$, we obtain $\epsilon_{\hat{F}}^{-1}(D'_{\lambda'}) \cap H_\ell = \emptyset$. Let B_ℓ be the component of $\check{F} \setminus F$ bounded by ℓ. Then B_ℓ is a once-punctured disk, and B_ℓ covers the once-punctured disk $Y_\ell \setminus f_{\check{F}}(p_\ell)$ via $f_{\check{F}}$. Since $Y_\ell \cap int(F') = \emptyset$, $B_\ell \cap \lfloor f_{\check{F}}^{-1}(\lambda') \rfloor = \emptyset$. Thus it suffices to show that $\epsilon_F^{-1}(D'_{\lambda'})$ is a multidisk properly embedded in H_F bounded by $\lfloor f_F^{-1}(\lambda') \rfloor$.

By Proposition 5.1 (i), $\epsilon_F^{-1}(D'_{\lambda'})$ is a union of finitely many (disjoint) disks of $\tilde{\Delta} \cap H_F$. By Proposition 5.1 (iii), a disk D of $\tilde{\Delta} \cap H_F$ embeds into $D'_{\lambda'}$ if and only if $f_{\check{F}}(\partial D) = f(\partial D) = \lambda'$. This completes the proof. $\square$

For a surface Σ, we let P_Σ denote the set of all punctures of Σ. Let $L_{\lambda'} = \lfloor f_{\check{F}}^{-1}(\lambda') \rfloor$, which is a multiloop on $\check{F}$.

Lemma 6.10. *Let* X *be a component of* $\check{F} \setminus L_{\lambda'}$. *Then* $C_{\check{F}}(X)$ *is an almost good genus-zero surface supported on a 2-disk with (finitely many) punctures, where the 2-disk is a component of* $\hat{\mathbb{C}} \setminus \lambda'$.

Proof. Since $L_{\lambda'} = \partial[\epsilon_{\hat{F}}^{-1}(D'_{\lambda'})]$ by Lemma 6.9, $\epsilon_{\hat{F}}$ embeds X into a single component H of $\overline{\mathbb{H}^3} \setminus D'_{\lambda'}$. Then $H \cap \hat{\mathbb{C}}$ is a component of $\hat{\mathbb{C}} \setminus \lambda'$, which is a round 2-disk. Recall that, if $p \in P_{\check{F}}$, in particular if $p \in P_X$, then $f_{\check{F}}(p) = \epsilon_{\hat{F}}(p)$; therefore, different points of P_X map to different points in $int(H \cap \hat{\mathbb{C}})$. In addition, all boundary components of X cover λ' via $f_{\check{F}}$. Therefore $C_{\check{F}}(X)$ is an almost good genus-zero surface fully supported on the punctured disk $(H \cap \hat{\mathbb{C}}) \setminus f_{\check{F}}(P_X)$. $\square$

Step 3. Let $\phi\colon \overline{\mathbb{H}^3} \to \overline{\mathbb{H}^3}$ be a homeomorphism. Then, by post-composing with ϕ, we can transform $C_{\check{F}}, \epsilon_{\hat{F}}, \eta_{\check{F}}$ without lose of their topological properties and correspondences (so that we can easily observe that $\lfloor f^{-1}(\mu') \rfloor$ is a single loop on $\check{F}$); namely, we let

$$\begin{aligned} f_\phi &= \phi \circ f_{\check{F}}\colon \check{F} \to \hat{\mathbb{C}} \\ C_\phi &= (f_\phi, \rho_{id}) \\ \epsilon_\phi &= \phi \circ \epsilon_{\hat{F}}\colon H_{\hat{F}} \to \overline{\mathbb{H}^3} \\ \eta_\phi &= \phi \circ \eta_{\check{F}}\colon \check{F} \times [0,1] \to \overline{\mathbb{H}^3}. \end{aligned}$$

Proposition 6.11. *There exists a homeomorphism $\phi\colon \overline{\mathbb{H}^3} \to \overline{\mathbb{H}^3}$ such that*
(i) $\lfloor f_\phi^{-1}(\lambda') \rfloor$ is a single loop isotopic to λ on $\check{F}$, and
(ii) λ' is isotopic to $\epsilon_\phi(\lambda)$ in $Im(\eta_\phi)$.

In the following, we will reduce the proof of Proposition 6.11 to an induction (Lemma 6.12). Consider the following condition:
(II) $\epsilon_\phi^{-1}(D'_{\lambda'})$ is a multidisk properly embedded in $H_{\hat{F}} \setminus P_{\check{F}}$ and bounded by $\lfloor f_\phi^{-1}(\lambda') \rfloor$.

Assume that there is a homeomorphism $\phi : \overline{\mathbb{H}^3} \to \overline{\mathbb{H}^3}$ satisfying (i) and (II). Then, by (i), $\lfloor f_\phi^{-1}(\lambda') \rfloor$ is a loop on $\check{F}$ isotopic to λ. Therefore $\epsilon_\phi(\lambda)$ is isotopic to $\epsilon_\phi(\lfloor f_\phi^{-1}(\lambda') \rfloor)$ on $\epsilon_\phi(\check{F}) = \eta_\phi(\check{F} \times \{0\})$. By (II), $D'_{\lambda'} \cap Im(\epsilon_\phi)$ is a single disk bounded by $\epsilon_\phi(\lfloor f_\phi^{-1}(\lambda') \rfloor)$. Therefore, $\epsilon_\phi(\lfloor f_\phi^{-1}(\lambda') \rfloor)$ and λ' bound an annulus properly embedded in $Im(\eta_\phi)$. Then, in particular, they are isotopic in $Im(\eta_\phi)$, and (ii) holds. Thus it suffices to construct ϕ satisfying (i) and (II).

As an induction hypothesis, we suppose that there is a homeomorphism $\phi_1\colon \overline{\mathbb{H}^3} \to \overline{\mathbb{H}^3}$ satisfying the following conditions:
(I) $\lfloor f_{\phi_1}^{-1}(\lambda') \rfloor =: L_{\phi_1}$ is a multiloop on $\check{F}$ containing a loop λ_{ϕ_1} isotopic to λ,
(II) $\epsilon_{\phi_1}^{-1}(D'_{\lambda'}) =: \Delta_{\phi_1}$ is a multidisk properly embedded in $H_{\hat{F}} \setminus P_{\check{F}}$ and bounded by L_{ϕ_1}, and
(III) if X is a component of $\check{F} \setminus L_{\phi_1}$, then $C_{\phi_1}|X$ is an almost good genus-zero surface fully supported on the punctured disk $B_X \setminus f_{\phi_1}(P_X)$, where B_X is the component of $\hat{\mathbb{C}} \setminus \lambda'$ containing $f_{\phi_1}(P_X)$.

Note that, if $\phi_1 = id$, then ϕ_1 satisfies (I), (II), (III) by Lemma 6.9 and Lemma 6.10.

A loop ℓ of a multiloop N on a surface Σ is called *outermost* if ℓ is a separating loop and a component of $\Sigma \setminus \ell$ contains no loops of N.

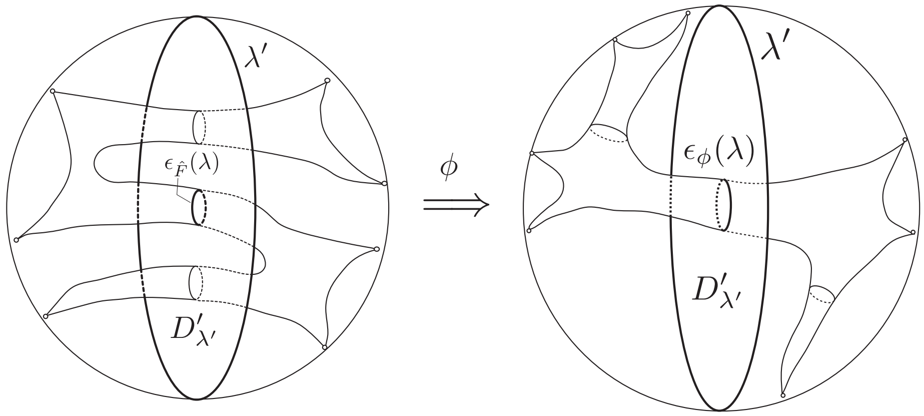

FIGURE 9. A basic example of ϕ realizing Proposition 6.11.

Such a component of $\Sigma \setminus \ell$ is called an *outermost component*. Note that every loop of L_{ϕ_1} on $\check{F}$ is separating since $\check{F}$ is a planar surface.

Lemma 6.12. *Let ℓ be an outermost loop of L_{ϕ_1} on $\check{F}$. Then there is a homeomorphism $\phi_2\colon \overline{\mathbb{H}^3} \to \overline{\mathbb{H}^3}$ satisfying the following properties:*
(I') $\lfloor f_{\phi_2}^{-1}(\lambda') \rfloor =: L_{\phi_2}$ is isotopic to $L_{\phi_1} \setminus \ell$ on $\check{F}$,
(II') $\epsilon_{\phi_2}^{-1}(D'_{\lambda'}) =: \Delta_{\phi_2}$ is a multidisk properly embedded in $H_{\hat{F}} \setminus P_{\check{F}}$ and bounded by L_{ϕ_2}, and
(III') if X is a component of $\check{F} \setminus L_{\phi_2}$, then $C_{\phi_2}|X$ is an almost good genus-zero surface supported on the punctured disk $B_X \setminus f_{\phi_2}(P_X)$, where B_X is the component of $\hat{\mathbb{C}} \setminus \lambda'$ containing $f_{\phi_2}(P_X)$.

This lemma implies Proposition 6.11:

Proof of Proposition 6.11 with Lemma 6.12 assumed. Note that Conclusions (I'), (II'), (III') on ϕ_2 correspond to Assumptions (I), (II), (III) on ϕ_1. Therefore, starting from the base case that $\phi_1 = id$, we repeatedly apply Lemma 6.12 and inductively reduce the number of the loops of L_{ϕ_1}. This reduction process preserves the loop isotopic to λ_{ϕ_1}. Thus, there is exactly one loop isotopic to λ left, we obtain ϕ satisfying (i), (II), as the composition of ϕ_2's obtained from this repeated application of Lemma 6.12. Hence this ϕ realizes Proposition 6.11 (as discussed above). $\square$

Proof (Lemma 6.12). First, we shall construct a homeomorphism ψ : $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\psi \circ \phi_1|_{\hat{\mathbb{C}}}\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a homeomorphism satisfying (I')

and (III'). Later, we extend ψ to a homeomorphism from $\overline{\mathbb{H}^3}$ to itself, such that $\psi \circ \phi_1 \colon \overline{\mathbb{H}^3} \to \overline{\mathbb{H}^3}$ satisfies (II') as well.

Let D_ℓ be the disk of Δ_{ϕ_1} bounded by ℓ. Let Q be the outermost component of $\check{F} \setminus L_{\phi_1}$ bounded by ℓ. Let H_Q be the closure of the component of $H_{\hat{F}} \setminus \tilde{\Delta}$ bounded by Q such that $H_Q \cap \hat{F} = Q \cup \ell$. By Assumption (II), $\epsilon_{\phi_1}(H_Q)$ is contained in the closure of a component of $\overline{\mathbb{H}^3} \setminus D'_{\lambda'}$. Let R be the component of $\check{F} \setminus L_{\phi_1}$ adjacent to Q, sharing ℓ as a boundary component. Then ∂R is a multiloop bounding a multidisk consisting of disks of Δ_{ϕ_1}. Let H_R be the closure of the component of $H_{\hat{F}} \setminus \Delta_{\phi_1}$ bounded by this multidisk, so that $H_R \cup \hat{F} = R$.

As a convention, we identify $H_{\hat{F}}$ with $\epsilon_\phi(H_{\hat{F}})$ via ϵ_{ϕ_1}. Then, D_λ is contained in $D'_{\lambda'}$, and (the interiors of) H_Q and H_R are contained in the different components of $\overline{\mathbb{H}^3} \setminus D'_{\lambda'}$. Our gold is to isotope $H_Q \cup H_R$ to the single component of $\overline{\mathbb{H}^3} \setminus D'_{\lambda'}$ containing H_R, while fixing $H_{\hat{F}} \setminus (H_Q \cup H_R)$. We will see that this isotopy will realize the desired homeomorphism ψ.

Let $B(\lambda', +)$ and $B(\lambda', -)$ be the components of $\hat{\mathbb{C}} \setminus \lambda'$. Since Q and R are adjacent, we can assume that $Supp(C_{\phi_1}|Q) = B(\lambda', +) \setminus P_Q$ and $Supp(C_{\phi_1}|R) = B(\lambda', -) \setminus P_R$ (by the convention above, for example, $\mathcal{P}_Q$ here means $\epsilon_{\phi_1}(P_Q) \subset \hat{\mathbb{C}}$). Since $Supp(C_{\phi_1}|Q)$ and $Supp(C_{\phi_1}|R)$ are punctured disks, by the definition of an almost good genus-zero surface, both Q and R must have at least one puncture. Set $P_Q = \{q_1, q_2, \dots, q_h\}$. Choose a round circle λ'' in the interior of $B(\lambda', -) \setminus P_{\check{F}}$ such that λ'' is isotopic to $\lambda' = \partial B(\lambda', -)$. Let A' be the annulus in $B(\lambda', -) \setminus P_{\check{F}}$ bounded by λ' and λ''. Similarly, let $B(\lambda'', +)$ and $B(\lambda'', -)$ be the components of $\hat{\mathbb{C}} \setminus \lambda''$ so that $B(\lambda'', +) = B(\lambda', +) \cup A'$ and $B(\lambda'', -) = B(\lambda', -) \setminus A'$.

Let $a_1, a_2, \dots, a_h$ be disjoint paths on the punctured disk $B(\lambda'', +) \setminus P_{\check{F}}$ such that a_i connects the point q_i to a point r_i in $int(A')$ for each $i \in \{1, 2, \dots, h\}$. We can in addition assume that each a_i transversally intersects λ' in a single point. Pick a 2-disk neighborhood U_i of a_i in $B(\lambda'', +)$ such that U_i contains no points of $P_{\check{F}}$ except q_i and U_i intersects λ' in a single arc. We can in addition assume that $U_1, U_2, \dots, U_h$ are disjoint. For each $i \in \{1, 2, \dots, h\}$, choose a homeomorphism $\sigma_i \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ supported on U_i such that $\sigma_i(q_i) = r_i$ (i.e. σ_i is the identity map on $\hat{\mathbb{C}} \setminus U_i$). Let $\psi = \sigma_1 \circ \sigma_2 \circ \dots \circ \sigma_h \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Then ψ is a homeomorphism supported on $\sqcup_{i=1}^h U_i =: U$. Note that there is an isotopy ξ_ψ of $\hat{\mathbb{C}}$ supported on U connecting ψ and the identity map. The restriction of f_{ϕ_1} to $\check{F} \setminus f_{\phi_1}^{-1}(P_{\check{F}})$ is a covering map from $\check{F} \setminus f_{\phi_1}^{-1}(P_{\check{F}})$ onto $\hat{\mathbb{C}} \setminus P_{\check{F}}$. Via this covering map, A' lifts to

finitely many disjoint cylinders $A_1, A_2, \dots, A_J$ in $\check{F}$. Therefore, letting $L^-_{\phi_1} = \lfloor f^{-1}_{\phi_1}(\lambda'') \rfloor$, each $A_j\,(j = 1, 2, \dots, J)$ is bounded by a loop of L_{ϕ_1} and a loop of $L^-_{\phi_1}$. Then, up to the cylinders $A_1, A_2, \dots, A_J$, the components of $\check{F} \setminus L_{\phi_1}$ are bijectively identified with the components of $\check{F} \setminus L^-_{\phi_1}$. By Assumption (III), if X is a component of $\check{F} \setminus L_{\phi_1}$, then $C_{\phi_1}|X$ is an almost good genus zero surface fully supported on either $B(\lambda', +) \setminus P_X$ or $B(\lambda', -) \setminus P_X$. Accordingly, if X is a component of $\check{F} \setminus L^-_{\phi_1}$, then $C_{\phi_1}|X$ is a good genus-zero surface supported on either $B(\lambda'', +) \setminus P_X$ (Case 1) or $B(\lambda'', -) \setminus P_X$ (Case 2). Since Q is outermost, Q has exactly one boundary component, which is ℓ. Then there exists $j \in \{1, 2, \dots, J\}$ such that A_j adjacent to Q, sharing ℓ as a boundary component. Then $Q \cup A_j$ is a component of $\check{F} \setminus L^-_{\phi_1}$, and it belongs to Case 1.

For each component X of $\check{F} \setminus L^-_{\phi_1}$, we shall see the difference between $L_{\phi_1} \cap X$ and $L_{\phi_2} \cap X$. In Case 2, since λ' and $Supp(C_{\phi_1}|X)$ are disjoint, by Lemma 4.3, $L_{\phi_1} \cap X = \emptyset$. Since ψ is the identity map on $Supp(C_{\phi_1}|X)$, we have $L_{\phi_2} \cap X = \emptyset$.

In Case 1, $A' \cap X$ is a regular neighborhood of ∂X, and it is a union of some A_j's. Then $X \cap L_{\phi_1}$ is a multiloop on X isotopic to ∂X. First, suppose that X does *not* contain Q. Then, since U is disjoint from $P_{\check{F}} \setminus P_Q$, P_X is disjoint from U. Thus $Supp(C_{\phi_1}|X)$ contains U. Since the isotopy ξ_ψ of $\hat{\mathbb{C}}$ is supported on U, lifting ξ_ψ via $dev(C_{\phi_1}|X)$, we obtain an isotopy from $X \cap L_{\phi_1}$ to $X \cap L_{\phi_2}$ on X.

Next suppose that $X = Q \cup A_j$ for some $j \in \{1, 2, \dots, J\}$. Then $X \cap L_{\phi_1} = \ell$ and ψ moves $P_X \subset B(\lambda', +)$ to $int(A') \subset B(\lambda', -)$. Thus $C_{\phi_2}|X$ is a good genus-zero surface fully supported on $B(\lambda'', +) \setminus \psi(P_X)$, and $B(\lambda'', +) \setminus \psi(P_X)$ contains $B(\lambda', +)$. Since $B(\lambda', +)$ is a disk bounded by λ', by Lemma 4.3, $X \cap L_{\phi_2} = \emptyset$. Combining all the cases above, we conclude that L_{ϕ_2} is isotopic to $L_{\phi_1} \setminus \ell$ on $\check{F}$ ((I')).

Via the isotopy between L_{ϕ_2} and $L_{\phi_1} \setminus \ell$, a component X of $\check{F} \setminus L_{\phi_2}$ is isotopic to either $Q \cup R$ or a component of $\check{F} \setminus L_{\phi_1}$ that is *not* Q or R. In either case, each boundary component of X covers λ' via $dev(C_{\phi_2}|X) = f_{\phi_2}|_X$. First, suppose that X is isotopic to $Q \cup R$. Then $\psi(P_Q) \subset A'$ and $\psi(P_R) = P_R \subset B(\lambda'', -)$. Therefore $dev(C_{\phi_2}|X) \colon X \to \overline{\mathbb{H}^3}$ takes all points of P_X to distinct points in $B(\lambda', -)$ and all boundary components of X to λ'. Thus $C_{\phi_2}|X$ is an almost good genus-zero surface whose support is $B(\lambda', -) \setminus \psi(P_X)$. Next, suppose that X is isotopic to a component Y of $\check{F} \setminus L_{\phi_1}$. Then ψ fixes P_X, and thus P_X is contained in either $B(\lambda', +)$ or $B(\lambda', -)$. Therefore $C_{\phi_2}|X$ is an almost good genus-zero surface whose support is, accordingly, $B(\lambda', +) \setminus \psi(P_X)$ or

$B(\lambda', -) \setminus \psi(P_X)$. (Moreover it is easy to see that $C_{\phi_2}|X = C_{\phi_1}|Y$ since a projective structure is defined up isotopy of its base surface.) Thus (III') holds.

Next we shall extend the homeomorphism $\psi\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ to a homeomorphism $\psi\colon \overline{\mathbb{H}^3} \to \overline{\mathbb{H}^3}$. Let $H'(\lambda', +) = Conv(B(\lambda', +))$ and $H'(\lambda', -) = Conv(B(\lambda', -))$, so that $\overline{\mathbb{H}^3} \setminus D'_{\lambda'} = H'(\lambda', +) \sqcup H'(\lambda', -)$. Similarly, let $H'(\lambda'', +) = Conv(B(\lambda'', +))$ and $H'(\lambda'', -) = Conv(B(\lambda'', -))$, so that $\overline{\mathbb{H}^3} \setminus Conv(\lambda'') = H'(\lambda'', +) \sqcup H'(\lambda'', -)$.

For each $i \in \{1, 2, \ldots, h\}$, let V_i be a closed 3-disk neighborhood of a_i in $H'(\lambda'', +)$ satisfying the following (regularity) conditions:
(i) $V_i \cap \hat{\mathbb{C}} = U_i$,
(ii) $V_i \cap D'_{\lambda'}$ is a 2-disk,
(iii) $V_i \cap H_{\hat{F}} =: T_i$ is a 3-disk, and
(iv) $V_1, V_2, \ldots, V_h$ are disjoint
(see Figure 10). By (ii), $\partial(V_i) \cap D'_{\lambda'} \cong \mathbb{S}^1$ is a union of an arc properly embedded in $D'_{\lambda'}$ and the arc $U_i \cap \lambda'$. By (i) and (iii), $\partial V_i \cap \mathbb{H}^3 = \partial V_i \setminus U_i$ is an open 2-disk properly embedded in $\mathbb{H}^3$, and it intersects $H_{\hat{F}}$ in a 2-disk K_i separating q_i and $P_{\check{F}} \setminus \{p_i\}$ in $H_{\hat{F}}$. Then we have $T_i \cap \partial V_i = K_i \cup q_i$.

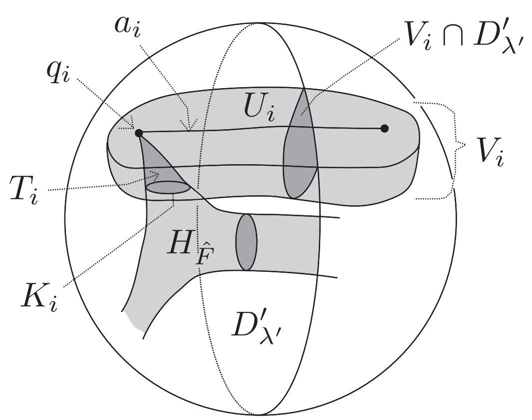

Figure 10. A picture of V_i.

Then, for each $i = 1, 2, \ldots, h$, we can extend the homeomorphism $\sigma_i\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ supported on U_i to a homeomorphism $\sigma_i\colon \overline{\mathbb{H}^3} \to \overline{\mathbb{H}^3}$ supported on V_i such that $\sigma_i(T_i)$ intersects $D'_{\lambda'}$ transversally in a single disk D_{T_i} (see (i) and (ii) in Figure 11). Accordingly, let $\sigma = \sigma_1 \circ \sigma_2 \circ \ldots \circ \sigma_h\colon \overline{\mathbb{H}^3} \to \overline{\mathbb{H}^3}$. Note that the isotopy ξ_ψ of $\hat{\mathbb{C}}$ also extends to an isotopy of $\overline{\mathbb{H}^3}$ supported on $V = \sqcup_{i=1}^h V_i$, connecting σ and the identity map.

From now, we regard $H_{\hat{F}}$ as a subset of $\overline{\mathbb{H}^3}$ via $\sigma \circ \epsilon_{\phi_1}$. (For instance, the puncture q_i of $\check{F}$ is identified with $r_i \in \hat{\mathbb{C}}$.) Since σ is a homeomorphism, $(\eta_\sigma :=)\ \sigma \circ \eta_{\check{F}} \colon \check{F} \times [0,1] \to \sigma \circ Im(\eta_{\check{F}}) \subset \overline{\mathbb{H}^3}$ is a homeomorphism, under which $\check{F} \times \{0\}$ maps to $\check{F}$ (i.e. $\sigma \circ \epsilon_{\hat{F}}(\check{F})$) and $\check{F} \times \{1\}$ maps to $\hat{\mathbb{C}} \setminus P_{\check{F}}$ (i.e. $\hat{\mathbb{C}} \setminus \sigma \circ \epsilon_{\hat{F}}(P_{\check{F}})$). Choose a puncture r of R. Let $\beta_1, \beta_2, \ldots, \beta_h$ be disjoint (open) paths on $\check{F} \subset H_{\hat{F}} \subset \overline{\mathbb{H}^3}$ such that β_i connects $q_i = r_i$ to r for each $i = 1, 2, \ldots, h$. In addition, we can assume that each β_i is disjoint from T_j if $i \neq j$ and it transversally intersects ∂K_i and $D'_{\lambda'}$, realizing minimal geometric intersection numbers in their isotopy classes. (Recall that D_ℓ is the disk of Δ_{ϕ_1} bounded by the outermost loop ℓ of L_{ϕ_1}, and it is embedded in $D'_{\lambda'}$.) Then $\beta_i \subset Q \cup R$, and β_i transversally intersects ∂K_i exactly in one point and $D'_{\lambda'}$ in two points, one point of ∂D_ℓ and one point of ∂D_{T_i} (see Figure 11 (ii)). Note that $\eta_\sigma(\beta_i \times [0,1]) \cup \{r_i, r\} =: E_i$ is a 2-disk properly embedded in $Im(\eta_\sigma) \cup P_{\check{F}}$. Then $E_1, E_2, \ldots, E_h$ share the point r, and $E_1 \setminus \{r\}, E_2 \setminus \{r\}, \ldots, E_h \setminus \{r\}$ are disjoint. In addition, we can assume that each E_i transversally intersects $D'_{\lambda'}$, if necessarily, by a small isotopy of E_i in $Im(\eta_\sigma)$. Then there is a unique arc component of $E_i \cap D'_{\lambda'}$ connecting the two intersection points of β_i and $D'_{\lambda'}$. Let e_i denote this arc component. Take disjoint (small) regular neighborhoods $N(E_1), N(E_2), \ldots, N(E_h)$ of $E_1 \setminus \{r, r_1\}, E_2 \setminus \{r, r_2\}, \ldots, E_h \setminus \{r, r_h\}$, respectively, in $Im(\eta_\sigma)$, such that $N(E_i) \cap V_j = \emptyset$ for all $i, j \in \{1, 2, \ldots, h\}$ with $i \neq j$. In addition, we can assume that $N(E_i) \cap D'_{\lambda'}$ is a regular neighborhood of $E_i \cap D'_{\lambda'}$ and that $N(E_i) \cap K_i$ is a single arc in ∂K_i. In particular, there is a component of $N(E_i) \cap D'_{\lambda'}$ that is a regular neighborhood of e_i in $D'_{\lambda'}$. Let $N(e_i)$ denote this regular neighborhood of e_i. Then $N(e_i)$ is a rectangular strip connecting D_ℓ and D_{T_i}, and $N(e_i) \cup D_{T_i}$ is a 2-disk properly embedded in the 3-disk $T_i \cup N(E_i) \cup \{r, r_i\}$. Take a small regular neighborhood of $N(e_i) \cup D_{T_i}$ in $T_i \cup N(E_i)$. Then identify this regular neighborhood with $(N(e_i) \cup D_{T_i}) \times [-1, 1]$ so that $(N(e_i) \cup D_{T_i}) \times [-1, 0) \subset H'(\lambda', -)$ and $(N(e_i) \cup D_{T_i}) \times (0, 1] \subset H'(\lambda', +)$. Then $N(e_i) \times \{-1\}$ is a 2-disk properly embedded in $N(E_i) \cong \mathbb{D}^3$, splitting $N(E_i)$ into two 3-disks. Let $N^+(E_i)$ denote the one of these two 3-disks containing $N(e_i) \times [-1, 1]$. Then $int(N^+(E_i))$ is disjoint from $int(H_{\hat{F}})$, and $\partial N^+(E_i) \cap \partial H_{\hat{F}}$ is a 2-disk contained in $Q \cup R$. Therefore we can isotope $H_{\hat{F}}$ to $H_{\hat{F}} \cup N^+(E_i)$ in $\overline{\mathbb{H}^3}$, fixing $H_{\hat{F}} \setminus (H_Q \cup H_R)$ and $\partial \mathbb{H}^3$. Now, we reidentify $H_{\hat{F}}$ with this $H_{\hat{F}} \cup N^+(E_i)$ $(i.e.\ \sigma \circ \epsilon_{\hat{F}}(H_{\hat{F}}) \cup N^+(E_i))$ via the isotopy.

Since $N^+(E_i)$ and T_i have disjoint interiors and their boundaries intersects in a 2-disk, $N^+(E_i) \cup T_i$ is a 3-disk. Then $(N(e_i) \cup D_{T_i}) \times \{-\frac{1}{2}\}$ is a 2-disk properly embedded in $N^+(E_i) \cup T_i$, and therefore it separates

$N^+(E_i) \cup T_i$ into two 3-disks. Let D^+ and D^- denote these two 3-disks so that D^+ contains $(N(e_i) \cup D_{T_i}) \times [-\frac{1}{2}, 1]$ and D^- contains $(N(e_i) \cup D_{T_i}) \times [-1, -\frac{1}{2}]$ (see Figure 11 (iii)). Then $D^+ \cap P_{\check{F}} = \emptyset$ and $D^- \cap P_{\check{F}} = \{q_i\}(= \{r_i\})$. Besides, $D^- \subset H'(\lambda', -)$ Note that D^+ and $H_{\hat{F}} \setminus D^+$ have disjoint interiors and their boundaries share a single 2-disk. Therefore we can isotope $H_{\hat{F}}$ to $H_{\hat{F}} \setminus D^+$ in $\overline{\mathbb{H}^3}$, fixing $H_{\hat{F}} \setminus (H_Q \cup H_R)$ and $\hat{\mathbb{C}}$ (see Figure 11 (iv)).

Let $\psi_i \colon \mathbb{H}^3 \to \mathbb{H}^3$ be the homeomorphism induced by the composition of all of the isotopies of $\overline{\mathbb{H}^3}$ that we have applied (so that ψ_i transforms Figure 11 (i) to (iv)). Then we shall compare this subset $H_{\hat{F}} \setminus D^+$ of $\mathbb{H}^3$ (i.e. $\xi_i \circ \epsilon_{\hat{F}}(H_{\hat{F}})$) with and the initial subset $H_{\hat{F}}$ of $\overline{\mathbb{H}^3}$ (i.e. $\epsilon_{\hat{F}}(H_{\hat{F}})$). Below we identify $H_{\hat{F}}$ with $\epsilon_{\hat{F}}(H_{\hat{F}})$, returning to the initial identification. Then we can see that the initial H_Q was transformed to $H_Q \setminus T_i$ and, letting $D_i^- = D^-$, the initial H_R to $H_R \cup D_i^-$. Topologically speaking, the marked point q_i on H_Q has just moved to H_R, by ψ_i. Note ψ_i fixes $H_{\hat{F}} \setminus (H_Q \cup H_R)$.

We can apply ψ_i simultaneously for all $i \in \{1, 2, \ldots, h\}$. Then H_Q is transformed to $H_Q \setminus (T_1 \cup T_2 \cup \ldots \cup T_h)$, which contains no points of $P_{\check{F}}$, and H_R is transformed to $H_R \cup (D_1^- \sqcup D_2^- \sqcup \ldots \sqcup D_h^-)$, which contains $P_Q \cup P_R$. We can see that $H_Q \setminus (T_1 \cup T_2 \cup \ldots \cup T_n)$ is topologically a 3-disk in $H'(\lambda', +)$ disjoint from $\hat{\mathbb{C}}$, its boundary intersecting $D'_{\lambda'}$ in a single 2-disk. Therefore, there is an isotopy of $\overline{\mathbb{H}^3}$ supported in small neighborhood of H_Q, such that this isotopy moves the entire subset $H_Q \cup int(H_R)$ into $H'(\lambda', -)$ and fixes $H_{\hat{F}} \setminus (H_Q \cup int(H_R))$. In particular, D_ℓ is contained in $H'(\lambda', -)$. Let $\psi \colon \overline{\mathbb{H}^3} \to \overline{\mathbb{H}^3}$ be the homeomorphism corresponding to the composition of all the isotopies of $\overline{\mathbb{H}^3}$ that we have applied to isotope $H_{\hat{F}} = \epsilon_{\hat{F}}(H_{\hat{F}})$. Then ψ fixes $H_{\hat{F}} \setminus (H_Q \cup H_R)$ and $(\phi \circ \epsilon_{\hat{F}})^{-1}(D'_{\lambda'}) =: \Delta_{\phi_2}$ is $\Delta_{\phi_1} \setminus D_\ell$.

Combining with (I'), $\partial \Delta_{\phi_2}$ is isotopic to L_{ϕ_2} on $\check{F}$. Modify ψ by post-composing with an isotopy of $\overline{\mathbb{H}^3}$ that fixes $\hat{\mathbb{C}}$ and, when restricted to $\check{F} \in \overline{\mathbb{H}^3}$, realizes the isotopy on $\check{F}$ between $\partial \Delta_{\phi_2}$ and L_{ϕ_2}. Then we have $\partial \Delta_{\phi_2} = L_{\phi_2}$ ((II')). $\square$

Proof (Proposition 6.8). Let $\phi \colon \overline{\mathbb{H}^3} \to \overline{\mathbb{H}^3}$ be the homeomorphism obtained by Proposition 6.11. By Proposition 6.11 (ii), there is an isotopy between λ' and $\epsilon_\phi(\lambda)$ in $Im(\eta_\phi)$. By the product structure of η_ϕ, there is also an isotopy between $\eta_\phi(\lambda \times \{0\}) = \epsilon_\phi(\lambda)$ and $\eta_\phi(\lambda \times \{1\})$ in $Im(\eta_\phi)$. Then there is an isotopy between λ' and $\eta_\phi(\lambda \times \{1\})$ in $Im(\eta_\phi)$. By postcomposing with ϕ^{-1}, we have an isotopy between $\phi^{-1}(\lambda')$ and $\phi^{-1} \circ \eta_\phi(\lambda \times \{1\}) = \eta_{\check{F}}(\lambda \times \{1\}) = \mu'$ in $\phi^{-1}(Im(\eta_\phi)) = Im(\eta_{\check{F}}) \cong$

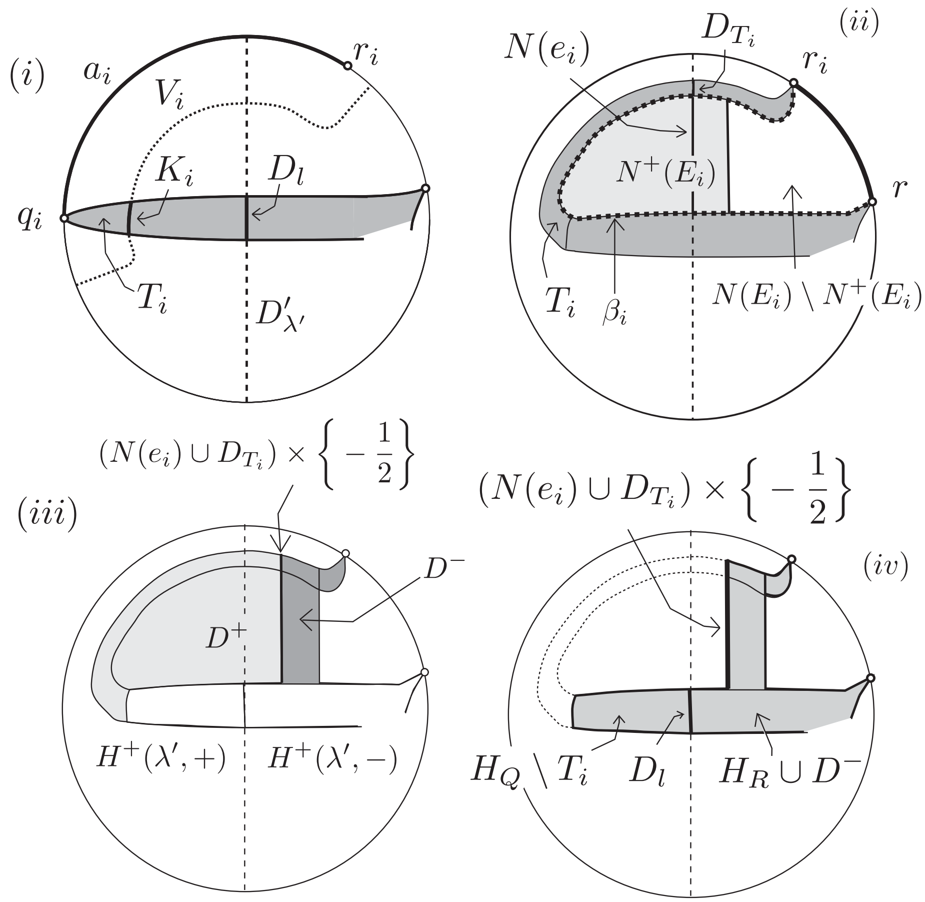

FIGURE 11. The series of isotopes of $H_{\hat{F}}$ in $\overline{\mathbb{H}^3}$.

$\check{F} \times [0,1]$. Note that $\phi^{-1}(\lambda')$ and μ' are loops on the punctured sphere $\hat{\mathbb{C}} \setminus \epsilon_{\hat{F}}(P_{\check{F}}) = \eta_{\check{F}}(\check{F} \times \{1\})$. By the canonical projection from $\check{F} \times [0,1]$ to $\check{F} \times \{1\}$, the isotopy between $\phi^{-1}(\lambda')$ and μ' in $Im(\eta_{\check{F}})$ induces to a homotopy and, therefore, an isotopy between $\phi^{-1}(\lambda')$ and μ' on $\hat{\mathbb{C}} \setminus \epsilon_{\hat{F}}(P_{\check{F}})$. Thus, via $f_{\check{F}}$, this isotopy between $\phi^{-1}(\lambda')$ and μ' lifts to an isotopy between the multiloops $f_{\check{F}}^{-1}(\phi^{-1}(\lambda')) = f_{\phi}^{-1}(\lambda')$ and $f_{\check{F}}^{-1}(\mu')$ on $\check{F}$. Therefore, their essential parts $\lfloor f_{\phi}^{-1}(\lambda') \rfloor$ and $\lfloor f_{\check{F}}^{-1}(\mu') \rfloor$ are also isotopic. By Proposition 6.11 (i), there is an isotopy between $\lfloor f_{\phi}^{-1}(\lambda') \rfloor$ and λ on $\check{F}$. Hence there is an isotopy between $\lfloor f_{\check{F}}^{-1}(\mu') \rfloor$ and λ on $\check{F}$. $\square$

7. A characterization of good structures by grafting

7.1. **A characterization of good punctured spheres.** Let F be a sphere with n punctures $p_1, p_2, \ldots, p_n$. Let $C = (f, \rho_{id})$ be a good projective structure on F. Then $f\colon F \to \hat{\mathbb{C}}$ continuously extends to a branched covering map $\hat{f}\colon F \cup \{p_1, p_2, \ldots, p_n\} \to \hat{\mathbb{C}}$. Let $\hat{F}$ denote $F \cup \{p_1, p_2, \ldots, p_n\}$, which is topologically a 2-shere. Since C is a good structure, $\hat{f}(p_1) =: q_1, \hat{f}(p_2) =: q_2, \ldots, \hat{f}(p_n) =: q_n$ are distinct points on $\hat{\mathbb{C}}$, and $Supp(C)$ is the n-punctured sphere $\hat{\mathbb{C}} \setminus \{q_1, q_2, \ldots, q_n\} =: R$. Choose a homeomorphism $f_0 : F \to R$ such that (its extension satisfies) $f_0(p_i) = q_i$ for all $i \in \{1, 2, \ldots, n\}$. Then (f_0, ρ_{id}) is a basic projective structure on F associated with C, where $\rho_{id}\colon \pi_1(F) \to \mathrm{PSL}(2, \mathbb{C})$ is the trivial representation (see §3.3, 3.4). Note that every basic projective structure on F associated with C can be obtained in such a way. We shall prove

Proposition 7.1. *Every good projective structure $C = (f, \rho_{id})$ on a punctured sphere F can be obtained by grafting a basic structure associated with C along a multiarc (each arc of which connects different punctures of F).*

For each $i \in \{1, 2, \ldots, n\}$, let d_i be the ramification index of $\hat{f}$ at the ramified point p_i. If $d_i > 1$, then p_i is called a **proper ramification point**. If $d_i = 1$, then p_i is called a **trivial ramification point**, i.e. f is a local homeomorphism at p_i. In the latter case, we regard p_i and q_i as marked points. Let d be the degree of $\hat{f}$, i.e. the cardinality of $\hat{f}^{-1}(x)$ for $x \in R$. Let $\delta = d - 1$ and $\delta_i = d_i - 1$ for each $i \in \{1, 2, \ldots, n\}$. Clearly we have $\delta \geq \delta_i \,(\geq 0)$ and, by the Riemann-Hurwitz formula, $2\delta = \Sigma_{i=1}^n \delta_i$ ($\in 2\mathbb{N}$). Therefore we have

$$2 \max_{1 \leq i \leq n} \delta_i \leq \sum_{i=1}^n \delta_i. \tag{1}$$

Lemma 7.2. *There is a multiarc A on F such that:*
(i) each arc of A connects distinct punctures of F, and
(ii) for each i, there are exactly δ_i arcs of A ending at p_i.

The following claim implies Lemma 7.2 (see also [10]):

Claim 7.3. *Let X be an n-gon. Set $e_1, e_2, \ldots, e_n$ to be the edges of X, listed in a cyclic order. For each $i = 1, 2, \ldots, n$, choose δ_i distinct marked points in the interior of e_i, i.e. e_i minus its end points. Then, there exists a multiarc A' properly embedded in X such that each arc*

of A' connects marked points on different edges of X and each marked point is an end of exactly one arc of A'.

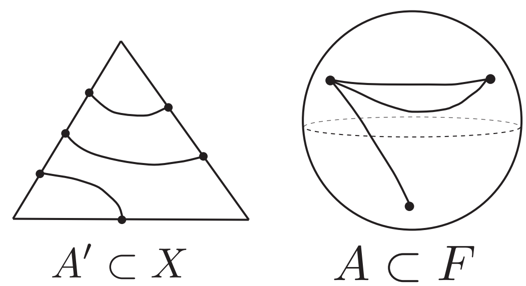

FIGURE 12. The multiarcs A and A' for $(\lambda_1, \lambda_2, \lambda_3) = (1, 2, 3)$.

Proof of Lemma 7.2 with Claim 7.3 assumed. For each $i \in \{1, 2, \ldots, n\}$, let e_i' be the closed arc contained in the interior of e_i such that e_i' contains all marked points on e_i. Consider the quotient space $X/\sim$ obtained by collapsing each e_i' to a single point, $e_i'/\sim$, so that the end points of A' on e_i are identified as being $e_i'/\sim$. Embed $X/\sim$ ($\cong \mathbb{D}^2$) into $\hat{F}$ so that $e_i'/\sim$ maps to p_i for each $i \in \{1, 2, \ldots n\}$. Then $A'/\sim$ (embedded in $\hat{F}$) realizes the desired multiarc A. $\square$

Proof of Claim 7.2. We prove this claim by induction on $\Sigma\delta_i \in 2\mathbb{N}$. As induction hypothesis, we assume that the lemma holds if $(\delta_1, \delta_2, \ldots, \delta_n)$ satisfies $\Sigma\delta_i = 2(k-1)$ for a fixed k. Now suppose that our $(\delta_1, \delta_2, \ldots, \delta_n)$ satisfies $\Sigma_{i=1}^n \delta_i = 2k$. Without loss of generality, we can assume that $\delta_1 = \max_{1 \le i \le n} \delta_i$. Let $m = \min\{i = 2, 3, \ldots, n \mid \delta_i \neq 0\}$. Then, let α be the arc properly embedded in X, connecting the marked point on e_1 closest to e_2 and the marked point on e_m closest to e_{m-1}.

Since a component of $X \setminus \alpha$ contains no marked points, it suffices to find a multiarc for the reduced n-tuple

$$\{\delta_1 - 1, 0, 0, \ldots, 0, \delta_m - 1, \delta_{m+1}, \delta_{m+2}, \ldots, \delta_n\} =: T,$$

which corresponds to the marked points contained in the other component of $X \setminus \alpha$. Then we have

$$(\delta_1 - 1) + 0 + \ldots + 0 + (\delta_m - 1) + \delta_{m+1} + \delta_{m+2} + \ldots + \delta_n = 2(k-1).$$

By Assumption (1), it is straightforward to show

$$2\, max\{\delta_1 - 1, 0, 0, \ldots, 0, \delta_m - 1, \delta_{m+1}, \delta_{m+2}, \ldots, \delta_n\} \le 2(k-1)$$

(by dividing it into the two cases that there are more than one i realizing $\max_{1 \le i \le n} \delta_i$ and that there are *not*). Therefore, by the induction

hypothesis, there is a multiarc A on X for T, so that A connects all marked points on e_i except for the end points of α and that A is disjoint from α. Then $\alpha \sqcup A$ is indeed the desired multiarc on X for the original n-tuple $\{\delta_1, \delta_2, \ldots, \delta_n\}$. $\square$

Proposition 7.4. *Let $f_1, f_2 \colon \hat{F} \to \hat{\mathbb{C}}$ be branched covering maps, such that, for each $i \in \{1, 2, \ldots n\}$ and $j \in \{1, 2\}$, p_i is a ramification point of f_j over q_i with the ramification index d_i and that f_j has no other proper ramification points. Then f_1 and f_2 are topologically equivalent, i.e. there are homeomorphisms $\phi \colon \hat{F} \to \hat{F}$ and $\phi' \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that*
(i) $\phi(p_i) = p_i$ and $\phi'(q_i) = q_i$ for all $i \in \{1, 2, \ldots, n\}$ and
(ii) $\phi' \circ f_1 = f_2 \circ \phi$.

Proof. Without loss of generality, we can assume that $d_1, d_2, \ldots, d_k > 1$ and $d_{k+1} = d_{k+2} = \ldots = d_n = 1$ for some integer $k \in \{1, 2, \ldots, n\}$. For each $j = 1, 2$, let $\mathcal{C}_j$ be the complex structure on $\hat{F} \cong \mathbb{S}^2$ obtained by pulling back the complex structure on $\hat{\mathbb{C}}$ via f_j. Then $f_j \colon (\hat{F}, \mathcal{C}_j) \to \hat{\mathbb{C}}$ is a meromorphic function. By the uniformization theorem, f_j is conformally equivalent to a rational function, i.e. there exist a rational function $\tau_j \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ and a conformal map $\psi_j \colon \hat{F} \to \hat{\mathbb{C}}$ such that $f_j = \tau_j \circ \psi_j$. Then, for each $i \in \{1, 2, \ldots, k\}$, $\psi_j(p_i)$ is the ramification point of τ_j over q_i with the ramification index d_i, and for each $i \in \{k+1, k+2, \ldots, n\}$, $\psi_j(p_i)$ is the trivial ramification point of τ_i.

By Theorem 3.2, there is a path τ_t ($t \in [1, 2]$) in $\mathcal{R}(d_1, d_2, \ldots, d_k)$ connecting τ_1 to τ_2 (see §3.6). Along τ_t, the (proper) ramification points of τ_t continuously move on the source sphere $\hat{\mathbb{C}}$ without hitting each other. Similarly the branched points of τ_t continuously move on the target sphere without hitting each other. Thus, for each $i \in \{1, 2, \ldots k\}$, there is a closed curve $Q_i(t)$ ($t \in [1, 2]$) on the target sphere such that $Q_i(1) = Q_i(2) = q_i$ and $Q_i(t)$ is a branched point of τ_t for all $t \in [1, 2]$. Then $Q_1(t), Q_2(t), \ldots, Q_n(t)$ are the branched points of τ_t for all $t \in [1, 2]$. Accordingly, for each $i \in \{1, 2, \ldots, k\}$, we have a (*not* necessarily closed) curve $P_i(t)$ ($t \in [1, 2]$) on the source sphere, such that $P_i(t)$ is the ramification point of τ_t over $Q_i(t)$ with the ramification index d_i for each $t \in [0, 1]$. Then $P_1(t), P_2(t), \ldots, P_k(t)$ are the branched points of τ_t for each $t \in [1, 2]$.

Note that $P_i(1) = \psi_1(p_i)$ and $P_i(2) = \psi_2(p_i)$ for each $i \in \{1, 2, \ldots, k\}$. For each $i \in \{k+1, k+2, \ldots, n\}$, pick a path $P_i(t)$ on $\hat{\mathbb{C}}$ connecting $\psi_1(p_i)$ to $\psi_2(p_i)$ so that $P_1(t), P_2(t), \ldots, P_n(t)$ are different points on the source sphere for each $t \in [1, 2]$. For each $i \in \{k+1, k+2, \ldots, n\}$, let $Q_i(t)$ ($t \in [1, 2]$) be a path on $\hat{\mathbb{C}}$ defined by $Q_i(t) = \tau_t(P_i(t))$. Then $Q_i(t)$ is a closed path starting at q_i. We can similarly assume that

$Q_1(t), Q_2(t), \ldots, Q_n(t)$ are different points on the target sphere for all $t \in [1, 2]$. In other words, we have an isotopy connecting the branched points of τ_1 and τ_2. Then this isotopy of the branched points extends to an isotopy of the target sphere $\xi'_t \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ $(t \in [1, 2])$. Recall τ_t is also continuous in t. Therefore, via τ_t, the isotopy ξ'_t on the target sphere lifts to an isotopy of the source sphere, $\xi_t \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ $(t \in [1, 2])$, so that $\tau_t \circ \xi_t = \xi'_t \circ \tau_1$ (first observe this lifting property locally using the local charts of the branched coverings τ_t). In particular we have $\tau_2 \circ \xi_2 = \xi'_2 \circ \tau_1$. Therefore we have

$$f_2 \circ (\psi_2^{-1} \circ \xi_2 \circ \psi_1) = \tau_2 \circ \xi_2 \circ \psi_1 = \xi'_2 \circ \tau_1 \circ \psi_1 = \xi'_2 \circ f_1.$$

Note that $\psi_2^{-1} \circ \xi_2 \circ \psi_1 \colon \hat{F} \to \hat{F}$ is a homeomorphism fixing p_i and that ξ'_2 is a homeomorphism of $\hat{\mathbb{C}}$ fixing q_i for all $i \in \{1, 2, \ldots, n\}$. □

Proof (Proposition 7.1). Let $C_0 = (f_0, \rho_{id})$ be a basic projective structure on F associated with C. Let A be the multiarc on F obtained by Lemma 7.2. Note that Lemma 7.2 (ii) is the condition for the multiarc in Proposition 7.1. Set $C_1 = (f_1, \phi_{id})$ to denote $Gr_A(C_0)$. (In the following, we conflate the developing map of a good projective structure on F and the branched covering map from $\hat{F}$ to $\hat{\mathbb{C}}$ obtained by continuously extending the developing map to the punctures of F.) Then, for all $i \in \{1, 2, \ldots, n\}$, we have $f(p_i) = f_1(p_i) = q_i$ and the ramification indices of f and f_1 are both d_i at p_i. Therefore, by Proposition 7.4, there are a homeomorphism $\phi \colon \hat{F} \to \hat{F}$ fixing all p_i and a homeomorphism $\phi' \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ fixing all q_i, such that $\phi' \circ f = f_1 \circ \phi$. Therefore $f = {\phi'}^{-1} \circ f_1 \circ \phi$. For a homeomorphism $\psi : F \to R$ and a multiarc N on F, let $Gr_N(\psi)$ denote the developing map of $Gr_N((\psi, \rho_{id}))$, where (ψ, ρ_{id}) is a basic projective structure on F. Then $f_1 = Gr_A(f_0)$. Therefore

$$f = {\phi'}^{-1} \circ Gr_A(f_0) \circ \phi = Gr_A({\phi'}^{-1} \circ f_0) \circ \phi = Gr_{\phi^{-1}(A)}({\phi'}^{-1} \circ f_0 \circ \phi).$$

Thus $({\phi'}^{-1} \circ f_0 \circ \phi,\ \rho_{id})$ is a basic projective structure on F, and $C = (f, \rho_{id})$ is obtained by grafting this basic projective structure along $\phi^{-1}(A)$ with the desired property. □

7.2. **A characterization of good holed spheres.** As an immediate corollary of Proposition 7.1, we obtain:

Proposition 7.5. *Let C be a good projective structure on a holed sphere F. Then C can be obtained by grafting a basic structure associated with C along a multiarc (each arc of which connects different boundary components of F).*

8. The proof of the main theorem

Recall that S is a closed orientable surface of genus g, Γ is a fuchsian Schottky group of rank g, and $\rho\colon \pi_1(S) \to \Gamma \subset \mathrm{PSL}(2,\mathbb{C})$ is an epimorphism.

Theorem 8.1. *Every Schottky structure $C = (f, \rho)$ on S can be obtained by grafting a basic Schottky structure with holonomy ρ (along a multiloop on S).*

Remark: By Lemma 3.1, a basic projective structure with holonomy ρ is Ω/Γ with some marking.

Proof. In §6, we constructed the multiloops M and M' on (S, C) and Ω/Γ, respectively. Recall also that $\tilde{M}$ and $\tilde{M}'$ are the total lifts of M and M' to $\tilde{S}$ and Ω, respectively. Set $(F_i, C_i)_{i=1}^n$ to be the components of $(S \setminus M, C(S \setminus M))$. By Theorem 6.1, (F_i, C_i) is a good holed sphere such that $Supp(C_i)$ is a component of $\Omega \setminus \tilde{M}'$ for each $i \in \{1, 2, \ldots, n\}$. By Proposition 7.5, each $C_i = Gr_{A_i}(C_{0,i})$, where $C_{0,i}$ is a basic structure on the holed sphere F_i with $Supp(C_{0,i}) = Supp(C_i)$ and A_i is a multiarc on F_i such that each arc of A_i connects distinct boundary components of F_i. For each loop ℓ of M, its lift $\tilde{\ell}$ covers a loop of $\tilde{M}'$ via f. Let d_ℓ be the degree of this covering map $f|_{\tilde{\ell}}$. The loop ℓ corresponds to exactly two boundary components of $F_1 \sqcup F_2 \sqcup \ldots \sqcup F_n$. Then, on each of these two boundary components, there are exactly $d_\ell - 1$ arcs of $A_1 \sqcup A_2 \sqcup \ldots \sqcup A_n$ ending. Therefore we can isotope A_i on F_i for all $i \in \{1, 2, \ldots, n\}$ so that the endpoints of $A_1, A_2, \ldots, A_n$ match up and $\cup A_i =: A$ is a multiloop on S.

Lemma 8.2. *(i) The union of $C_{0,i}$ on F_i (over $i = 1, 2, \ldots, n$) is a basic Schottky structure on S with holonomy ρ.*
(ii) For each loop α of A, $\rho(\alpha)$ is loxodromic, i.e. $\rho(a) \neq 1$.

Proof. (i). Assume that C_i and C_j are adjacent components of $C \setminus M$, sharing a boundary component ℓ. Then $Supp(C_i)$ and $Supp(C_j)$ are adjacent components of $\Omega \setminus \tilde{M}$ (up to an element of Γ), sharing a boundary component $f(\tilde{\ell})$, where $\tilde{\ell}$ is a lift of ℓ to $\tilde{S}$. Since $C_{0,i}$ and $C_{0,j}$ are the canonical projective structures on $Supp(C_i)$ and $Supp(C_j)$, respectively, we can pair up and identify the boundary components of $C_{0,i}$ and $C_{0,j}$ corresponding to ℓ. In such a way, we can identify all boundary components of $C_{0,i}$ $(i = 1, 2, \ldots, n)$ and obtain a projective structure C_0 on S. Let $\tilde{C}_0 = (f_0, \rho_{id})$ be the projective structure on $\tilde{S}$ obtained by lifting C_0 to $\tilde{S}$, where f_0 is a $\tilde{\rho}$-equivariant immersion from $\tilde{S}$ to $\hat{\mathbb{C}}$. Then, for each component R of $\tilde{S} \setminus \tilde{M}$, $f_0|_R$ is an embedding

onto $Supp(\tilde{C}|R)$, where $\tilde{C}$ is the projective structure on $\tilde{S}$ obtained by lifting C. By Theorem 6.1, there is a $\tilde{\rho}$-equivariant homeomorphism $\zeta\colon \tilde{S} \to \Omega$ such that $Supp(C|R) = \zeta(R)$ for each component R of $\tilde{S} \setminus \tilde{M}$ and that the restriction $f|_\ell$ is a covering map from ℓ onto $\zeta(\ell)$ for each loop ℓ of $\tilde{M}$. Thus $f_0|_R$ is a homeomorphism of R onto $\zeta(R)$, and $f_0|_\ell$ is a homeomorphism from ℓ onto $\zeta(\ell)$ for each loop ℓ of $\tilde{M}$. Therefore f_0 is a $\tilde{\rho}$-equivariant homeomorphism onto Ω, and (S, C_0) is a basic Schottky structure with holonomy ρ.

(ii). Let α be a loop of A. Then, set $\alpha = a_1 \cup a_2 \cup \ldots \cup a_m$, where $a_1, a_2, \ldots, a_m$ are different arcs of $A_1 \sqcup A_2 \sqcup \ldots \sqcup A_n$. Let $\tilde{\alpha}$ be a lift of α to $\tilde{S}$. Then each a_j $(j = 1, 2, \ldots, m)$ is an arc properly embedded in F_i with some $i \in \{1, 2, \ldots, n\}$, connecting different boundary components of F_i. Therefore, for each component P of $\tilde{S} \setminus \tilde{M}$, either $\tilde{\alpha}$ is disjoint from P or $\tilde{\alpha}$ intersects P in a single arc connecting different boundary components of P. Thus $\tilde{\alpha}$ is a biinfinite simple curve properly embedded in $\tilde{S}$, and (the homotopy class of) α translates $\tilde{S}$ along $\tilde{\alpha}$. Therefore $\rho(\alpha)$ is loxodromic. □

We will show that C is obtained by grafting the basic structure $C_0 = \cup_{i=1}^n C_{0,i}$ along A; the main step is to show

$$\cup_{i=1}^n Gr_{A \cap F_i}(C_{0,i}) = Gr_A(\cup_{i=1}^n G_{0,i}),$$

which means that the grafting Gr_A on C_0 "commutes" with the decomposition $C_0 = \cup_{i=1}^n G_{0,i}$.

For each $j \in \{1, 2, \ldots, m\}$, let b_j, c_j denote the boundary components of $C_{0,i} = (F_i, C_{0,i})$, with some $i \in \{1, 2, \ldots, n\}$, connected by a_j. Via $dev(C_{0,i})$, $C_{0,i}$ is isomorphic to $Supp(C_{0,i})$ in $\hat{\mathbb{C}}$, which is an component of $\Omega \setminus \tilde{M}'$. Via this isomorphism, b_j and c_j bound a projective cylinder Y_j in $\hat{\mathbb{C}}$, and Y_j contains $a_i\,(= dev(C_{0,i})(a_i))$ connecting the boundary components of Y_j. Then $Gr_{a_j}(C_{0,i})$ is obtained by appropriately identifying the boundary arcs of $Y_j \setminus a_j$ and $C_{0,i} \setminus a_j$ corresponding to a_j. Suppose that, for some $j_1, j_2 \in \{1, 2, \ldots, m\}$, a_{j_1} and a_{j_2} are adjacent arcs in a, sharing an endpoint $v \in M$. For each $k = 1, 2$, similarly, let i_k be the element of $\{1, 2, \ldots, n\}$ such that $a_{j_k} \subset C_{0,i_k}$ and also let Y_{j_k} be its corresponding projective cylinder. Then C_{0,i_1} and C_{0,i_2} are isomorphic to some adjacent components of $C_0 \setminus M$, sharing a boundary component containing v. Accordingly, Y_{j_1} and Y_{j_2} are also adjacent cylinders embedded in $\hat{\mathbb{C}}$ bounded by adjacent loops of $\tilde{M}'$, sharing a boundary component containing v. Thus we can identify the corresponding boundary components of Y_{j_1} and Y_{j_2}. Similarly, we identify all corresponding boundary components of Y_j $(j = 1, 2, \ldots, m)$

and obtain a projective torus T (we obtain a connected surface, since a is connected). Then T contains the loop $a = \cup_i a_i$.

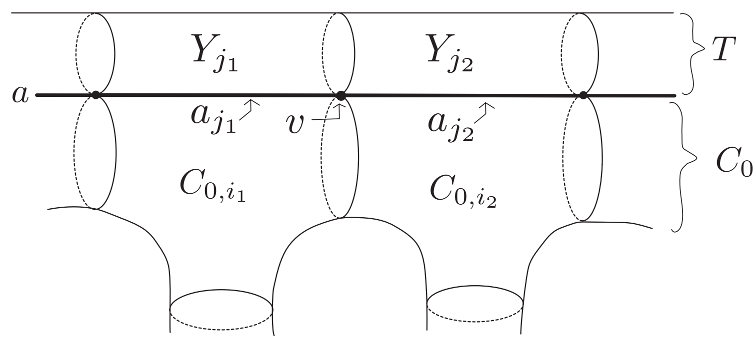

Figure 13.

We shall show that T is a hopf torus. Let N be the union of boundary components of Y_j, which is the multiloop on T that splits T into Y_j's. The homotopy class of a generates an infinite cyclic subgroup $\langle a \rangle$ of $\pi_1(S)$. Let $\tilde{T}$ denote the projective cylinder obtained by lifting T to its infinite cyclic cover whose covering transformation group is $\langle a \rangle$. Let $\tilde{N}$ denote the total lift of N to $\tilde{T}$. Then $\tilde{a} \subset \tilde{S}$ above can be isomorphically identified with the lift of a on T to $\tilde{T}$. Note that $\tilde{a}$ transversally intersects each loop of $\tilde{N}$. We shall show that $\tilde{T}$ is isomorphic to $\hat{\mathbb{C}} \setminus Fix(\rho(a))$, where $Fix(\rho(a))$ is the set of the two fixed points of $\rho(a)$ on $\hat{\mathbb{C}}$. Let m_h $(h \in \mathbb{Z})$ denote the loops of $\tilde{M}$ intersecting $\tilde{a} \subset \tilde{S}$ so that, for each $h \in \mathbb{Z}$, m_h and m_{h+1} are adjacent, i.e. they are boundary components of a single component of $\tilde{S} \setminus \tilde{M}$. Accordingly $\zeta(m_h)$ $(h \in \mathbb{Z})$ are the circles on $\hat{\mathbb{C}}$ that split $\hat{\mathbb{C}} \setminus Fix(\rho(a))$ into cylinders bounded by $\zeta(m_h)$ and $\zeta(m_{h+1})$. In addition, m_h's bijectively correspond to the loops of $\tilde{N}$ on $\tilde{T}$ via the identification of $\tilde{a} \subset \tilde{S}$ and $\tilde{a} \subset \tilde{N}$. Using this correspondence, we see that the components of $\tilde{T} \setminus \sqcup_h m_h$ are isomorphic to the components of $\hat{\mathbb{C}} \setminus (\, Fix(\rho(a)) \sqcup (\sqcup_h \zeta(m_h))\,)$. Thus $\tilde{T}$ is isomorphic to $\hat{\mathbb{C}} \setminus Fix(\rho(a))$. Therefore T is the quotient of $\hat{\mathbb{C}} \setminus Fix(\rho(a))$ by the cyclic group $\langle \rho(a) \rangle$, which is a Hopf torus.

To complete the proof, we now analyze the grafting operation along the entire multiloop A. Set $A = \alpha_1 \sqcup \alpha_2 \sqcup \ldots \sqcup \alpha_r$, where $\alpha_1, \alpha_2, \ldots, \alpha_r$ are the loops of A on S. By further decomposing each α_j $(j = 1, 2, \ldots, r)$, as in the proof Lemma 8.2 (ii), we set $A = a_1 \cup a_2 \cup \ldots \cup a_m$ so that $a_1, a_2, \ldots, a_m$ are the components of the multiarcs $A \cap F_1, A \cap F_2, \ldots, A \cap F_n$. Then, for each $j \in \{1, 2, \ldots, m\}$, a_j is an

arc properly embedded in $F_{i(j)}$ with some $i(j) \in \{1, 2, \dots, n\}$. Then, let Y_j denote the projective cylinder associated with $Gr_{a_j}(C_{0,i(j)})$, i.e. $Gr_{a_j}(C_{0,i(j)}) = (C_{0,i(j)} \setminus a_j) \cup (Y_j \setminus a_j)$. Then, for each $i \in \{1, 2, \dots, n\}$, we have

$$Gr_{A_i}(C_{0,i}) = (C_{0,i} \setminus A_i) \cup (\sqcup \{ Y_j \setminus a_j \mid a_j \subset A_i,\ j = 1, 2, \dots, m\}).$$

For each $k \in \{1, 2, \dots r\}$, let T_k denote the Hopf torus associated with α_k, which is $\cup \{Y_j \mid a_j \subset \alpha_k,\ j = 1, 2, \dots, m\}$. Then, we have

$$\begin{aligned} C &= \cup_{i=1}^n Gr_{A_i}(C_{0,i}) \\ &= \cup_{i=1}^n [\ (C_{0,i} \setminus A_i) \cup (\sqcup \{ Y_j \setminus a_j \mid a_j \subset A_i\})\ ] \\ &= [\cup_{i=1}^n (C_{0,i} \setminus A_i)] \sqcup [\cup_{j=1}^n (Y_j \setminus a_j)] \\ &= (C_0 \setminus A) \cup [(T_1 \setminus \alpha_1) \cup (T_2 \setminus \alpha_2) \cup \dots \cup (T_r \setminus \alpha_r)] \\ &= Gr_A(C_0)\ . \end{aligned}$$

□

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